Finite time optimizations of a Newton's law Carnot cycle

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We treat the problem of optimal finite time operations of a heat engine using an arbitrary working fluid and working between two constant temperature heat reservoirs. We work in a simplified framework ("Newton's law thermodynamics") which considers only losses associated with the heat exchange processes. We find the operations which maximize power, efficiency, effectiveness, and profit and those which minimize the loss of available work and the production of entropy. We find that all these optimal operations take place with the working fluid exchanging heat at a constant rate with each reservoir (implying a constant rate of entropy production) and undergoing adiabatic processes instantaneously. We define "Carnot space" to be the set of all operations of the engine which consist of constant rate heat exchanges and instantaneous adiabats. All optimal operations are points in this space which is shown (within the model) to be three dimensional. The different optimal operations with different connotations of "optimal" as described above are compared within this framework. To further study the economic implication of this model we also view the operation of the engine as an economic production process with work as its output. We obtain a simple analytical form of the production function and see repeatedly that maximum profit operation is a compromise between operation which maximizes the power and operation which minimizes the loss of available work. The path of maximum profit is obtained as a function of the costs of power and of availability.

I. INTRODUCTION

The optimal operation of heat engines in finite time has been a subject of several recent discussions.[1-8] A complete treatment has to take into account all loss mechanisms such as mechanical friction, heat leaks, heat resistance at the boundaries, and internal dissipative processes. So far, a much more limited model which takes into account only heat resistance losses has been studied. Recently, we have investigated the implication of an even simpler model,[1] which we refer to as Newton's law thermodynamics. In this model, thermodynamic systems which exchange heat and work are always in internal equilibrium, though they may or may not be in equilibrium with each other. Further, work exchange is taken to proceed reversibly and at arbitrary rates. The description of processes in Newton's law thermodynamics differs from a reversible description only with respect to heat exchanges, which takes place according to Newton's law of cooling,[9] i.e., at a rate which is proportional to the difference in the temperatures of the systems exchanging the heat. This adds time and irreversibility to the thermodynamic description, while still maintaining the possibility of obtaining some general results concerning the optimal operation. Formally, the model is defined by the following axioms:

(1) Heat flows between the working fluid of temperature T and the reservoir of temperature T^{ex} through a wall of heat conductance κ at a rate

$$dQ/dt = \kappa(T - T^{ex}) , \quad (1.1)$$

where dQ/dt is the heat flux across the wall.

(2) The working fluid is in internal equilibrium at all times. Cycles for which this assumption holds have been called *endoreversible*, i.e., internally reversible. Physically, this assumes that the internal relaxation times of the working fluid are short compared to the time scale of the cycle.

(3) No friction, i.e., the transfer of work to and from the working fluid, proceeds reversibly.

(4) Mechanical coupling to the working fluid is free from inertial effects. Thus, work can enter or leave the working fluid at arbitrary rates.

Note that axioms (3) and (4) imply that reversible adiabatic processes can proceed in zero time.

A fundamental problem in the theory of the optimal operation of finite time thermodynamic systems concerns the choice of objective function, i.e., the function of merit which should be extremal for the optimal cycle. One may choose, for example, power, efficiency, or, with economic factors considered, cost or profit. Obviously, a theory of optimal operation should have the objective function as one of its fundamental variables.

In this paper we investigate the optimal operation of the Carnot cycle in Newton's law thermodynamics for different choices of the objective functions. A *Carnot engine* is *defined* as an engine which works between two heat reservoirs of constant temperatures T_1^{ex} and T_2^{ex}. Several authors[3-6] have solved the problem of maximum power in such engines. More recently, Rubin[4] has solved the problem of maximum efficiency, and Salamon, Nitzan, Andresen, and Berry[1] have solved the problem of minimum loss of availability. In line with these results we show that in Newton's law thermodynamics all optimal Carnot cycles are of the Curzon–Ahlborn[6] type, i.e., heat exchange branches take place with the *working fluid* at a constant temperature. Further, we show that

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adiabatic branches take place instantaneously. This is true whether one is optimizing power, efficiency, effectiveness, loss of availability, entropy production, cost, or profit. Further, it is true for arbitrary working fluids, e.g., a gas in a cylinder or a paramagnetic salt in a magnetic field. Having shown these two facts, we will deduce that all optimal operations of a Carnot cycle in Newton's law thermodynamics lie in a three dimensional space which we will call Carnot space. Finally, we will plot the positions of the various optimal cycles in Carnot space. This will enable us to elucidate the relationships among the various economic and thermodynamic criteria of merit.

II. REDUCTION TO ONE BRANCH OPTIMIZATIONS

Consider a heat engine working between two heat reservoirs at constant temperatures T_1^{ex} and T_2^{ex}. Contact with the heat reservoirs occurs via heat conductance κ. We assume that our heat engine operates in finite time. Therefore, the temperature of the working fluid must be different from the temperature of the reservoir. We will optimize the temperature of the working fluid as a function of time during the heat exchange and will show that optimal operation of this heat engine takes place only if the *working fluid remains at a constant temperature during the heat exchange*. We will see that this holds for each of the objective functions which define the various connotations of optimal operation.

Consider first a single heat exchange branch of an optimal cycle. (By a single branch we mean the path traced out during the time evolution of the working fluid while it remains in uninterrupted contact with a heat reservoir.) Since variations in this branch produce feasible alternative operations of the cycle, the optimal cycle must be optimal with respect to such variations. Accordingly, we seek the Euler–Lagrange equations which express this fact in mathematical terms.

Thus, we seek the time behavior along the single branch that extremizes our objective function with the rest of the cycle remaining completely unaltered. We assume that the optimal cycle operates with a certain cycle time[10] $\tau=\tau_1+\tau_2+\tau_3+\tau_4$, where τ_1 and τ_2 are the times spent on the heat exchange branches 1 and 2, respectively, and τ_3 and τ_4 are the times spent on the adiabatic branches 3 and 4, respectively. Variations of one heat exchange branch do not alter any of the τ_i, $i=1, \ldots, 4$, the behavior of the working fluid on the other branches, or the initial and final states of the working fluid on the varied branch.

In Newton's law thermodynamics we are free to make instantaneous adiabatic jumps (see the note following axioms) anywhere along any heat exchange branch. Thus, the initial and final states of the branch under consideration are constrained only to lie on two given adiabats; their exact locations on these adiabats are not important. This in turn implies that the branch optimization should be done with a constrained change σ in the entropy of the working fluid during the branch.

Below we make liberal use of the principle of monotonic substitution,[11] which states that a function(al) $u(x)$ is extremal if and only if $g(u)$ is extremal for any monotonic function g. For differentiable g, this is obvious from the respective extremal conditions

$$0=\delta g(u(x))=\frac{dg}{du}\,\delta u(x)\ , \tag{2.1}$$

$$0=\delta u(x)\ , \tag{2.2}$$

and the fact that g monotonic implies $dg/du\neq 0$ for any u. If $dg/du>0$, maximum g corresponds to maximum u and minimum g corresponds to minimum u, while if $dg/du<0$, maximum g corresponds to minimum u and minimum g corresponds to maximum u.

We now express each of our objective functions as monotonic functions of the heats Q_1 and Q_2 absorbed by the working fluid on the heat exchange branches 1 and 2, respectively. This will enable us to apply the principle of monotonic substitution to conclude that for variations of branch i, these objective functions are extremal if and only if Q_i is extremal. Thus, we will see that for one branch variations, all our objective functions can be replaced by the heat exchanged on the branch as far as the optimization procedure is concerned. This does not mean that our engine will run the same whether we program it to optimize, e.g., efficiency or power. The different objective functions will require different cycle times, different time allocations for the two heat exchange branches, and the branches will run between different adiabats, i.e., correspond to different entropy changes σ. It does mean that each of the objective functions behaves along branches 1 and 2 so as to extremize Q_1 and Q_2 provided that the times τ_i spent along the branches and the entropy change σ of the working fluid are constrained to have their optimal values.

In what follows take $T_1^{ex}>T_2^{ex}$ and use the sign conventions $Q_1>0$, $Q_2<0$ and $W=Q_1+Q_2$, where $W>0$ is the work produced by our heat engine.

A. Case A: Maximum power

The power P delivered by our cycle is given by

$$P=\frac{W}{\tau}=\frac{Q_1+Q_2}{\tau}=g(Q_1,\ Q_2)\ . \tag{2.3}$$

Note that $\partial P/\partial Q_i=(1/\tau)>0$, so maximum power occurs for maximum Q_1 and maximum Q_2.

B. Case B: Maximum efficiency

For the efficiency we get

$$\eta=\frac{W}{Q_1}=1+\frac{Q_2}{Q_1}=g(Q_1,\ Q_2)\ , \tag{2.4}$$

$$\frac{\partial\eta}{\partial Q_1}=-\frac{Q_2}{Q_1^2}\ , \tag{2.5}$$

$$\frac{\partial\eta}{\partial Q_2}=\frac{1}{Q_1}\ . \tag{2.6}$$

With our sign conventions $Q_1>0$ and $Q_2<0$. Thus, both partial derivatives are positive and maximum efficiency occurs for maximum Q_1 and maximum Q_2.

C. Case C: Maximum effectiveness

Effectiveness ϵ is defined by

$$\epsilon = W/W_{rev} , \tag{2.7}$$

where W_{rev} is the work done by a reversible process which operates between the same initial and final states of the reservoirs, i.e., the change in the availability of the heat reservoirs. W_{rev} is generally defined[12] with respect to an environment, of temperature T_{en}, from (to) which the reversible process transfers heats $Q_1(T_{en}/T_1^{ex})$ and $Q_2(T_{en}/T_2^{ex})$ (i.e., it is the work generated by two independent reversible heat engines constructed to work between the environment and each of the heat reservoirs):

$$W_{rev} = Q_1\left(1 - \frac{T_{en}}{T_1^{ex}}\right) + Q_2\left(1 - \frac{T_{en}}{T_2^{ex}}\right) , \tag{2.8}$$

$$= Q_1 \eta_1 + Q_2 \eta_2 , \tag{2.9}$$

where η_i is the (constant) efficiency of a reversible engine operating between the environment and reservoir i. Note that if $T_2^{ex} = T_{en}$, this assumes the more familiar form

$$W_{rev} = Q_1\left(1 - \frac{T_2^{ex}}{T_1^{ex}}\right) , \tag{2.10}$$

$$= Q_1 \eta_{rev} . \tag{2.11}$$

Further,

$$\epsilon = \frac{W}{Q_1 \eta_{rev}} = \frac{\eta}{\eta_{rev}} , \tag{2.12}$$

and so for this case optimizing ϵ is equivalent to optimizing η. We now show that this holds generally. Letting $X = Q_2/Q_1$ we have

$$\epsilon = \frac{Q_1 + Q_2}{Q_1 \eta_1 + Q_2 \eta_2} = \frac{1 + X}{\eta_1 + \eta_2 X} , \tag{2.13}$$

while

$$\eta = (Q_1 + Q_2)/Q_1 = 1 + X . \tag{2.14}$$

Since

$$\frac{d\epsilon}{dX} = \frac{T_{en}(T_1^{ex} - T_2^{ex})}{T_1^{ex} T_2^{ex}(\eta_1 + \eta_2 X)^2} > 0 \tag{2.15}$$

and

$$d\eta/dX = 1 > 0 , \tag{2.16}$$

both ϵ and η are maximum provided $X = Q_2/Q_1$ is maximum. As we saw in case B above, this occurs for maximum Q_1 and maximum Q_2.

D. Case D: Minimum entropy production

ΔS_u, the total change in the entropy of the universe during a given process, is the sum of all entropy changes of the systems participating in the process. As the change in entropy for the working fluid and the work reservoir is zero for a cycle, this sum consists of only two terms, one for each heat reservoir:

$$S_u = \frac{-Q_1}{T_1^{ex}} + \frac{-Q_2}{T_2^{ex}} > 0 . \tag{2.17}$$

Therefore,

$$\frac{\partial \Delta S_u}{\partial Q_i} = \frac{-1}{T_i^{ex}} < 0 , \tag{2.18}$$

and minimum ΔS_u occurs for maximum Q_1 and maximum Q_2.

E. Case E: Minimum loss of availability

ΔA_u, the total change in the availability of the universe, is the sum of all the changes in the availabilities of systems participating in the process. For a cycle, this also vanishes for the working fluid. The sum thus consists of three terms: two for the heat reservoirs and one for the work reservoir:

$$\Delta A_u = (-\eta_1 Q_1) + (-\eta_2 Q_2) + W , \tag{2.19}$$

$$= \frac{T_{en} Q_1}{T_1^{ex}} + \frac{T_{en} Q_2}{T_2^{ex}} = -T_{en} \Delta S_u . \tag{2.20}$$

From the last expression we see that minimizing the loss of availability $-\Delta A_u = |\Delta A_u|$ is equivalent to minimizing the total entropy production ΔS_u,[13] which by case D above requires Q_1 and Q_2 to be maximum.

Before considering the economic objective functions profit, cost, and output, we review the rudiments of economic optimizations.

F. Economic optimizations

We view the operation of our engine as a production process with work as its output. We carry out the short-run optimization[14] explicitly, but the method may be applied in a similar manner to long-run optimizations, in which case the engine parameters κ and σ_{max} (defined below) may be varied.

Economic optimization of a production process requires three ingredients.[14] The first ingredient is the representation of the technology through the *production function* $Y = \mathcal{P}(X)$, which expresses the most product Y which can be obtained by the technology as a function of the quantities of inputs X. The other two ingredients represent the economic environment. They are the *cost function* $C(X)$, which expresses the costs incurred for given quantities of inputs to production, and the *revenue function* $R(Y)$, which expresses the money income from the sale of a given quantity of output Y. Below we assume perfect competition, in which case the prices for all inputs and outputs of our production process are given constants independent of our production process. This will happen, for example, if the firm is unable to alter the prevailing prices by altering the quantities it produces or consumes. This assumption implies that the cost and revenue functions are linear functions of X and Y, respectively. The form of the production function P will require some work to derive (see Sec. VIII).

Given these three ingredients, one usually considers three types of economic optimizations of a production process[15]: (1) maximum output for a constrained cost, (2) minimum cost for a given output, and (3) maximum profit $\pi = R - C$. The sets of optimal operations in these three cases coincide, although each optimization suggests different natural parameterizations of this set.[16] The natural parameters are (1) the prices of the inputs and the given budget constraint, (2) the prices of the inputs and the given output constraint, and (3) the prices of the inputs and the price of the output.

For now we consider only the profit maximization, since this optimization can be performed with our technique of using one branch variations. In Sec. VIII we will return briefly to the other economic optimizations.

G. Case F: Maximum profit

The profit π is calculated for some convenient accounting period. Choosing this accounting period as our unit of time, the number of cycles performed by the engine in this time period is $1/\tau$. The work per unit time produced by out engine is therefore (work per cycle) $\cdot$ (number of cycles) $= W/\tau$. If P_W is the price of work, we have

$$R = P_W \cdot W/\tau \,. \tag{2.21}$$

We assume that the only input to the production process is the availability A taken from the reservoirs. Per cycle

$$A = W_{rev} = \eta_1 Q_1 + \eta_2 Q_2 \,, \tag{2.22}$$

which corresponds to a cost (per unit time)

$$C = P_A \cdot A/\tau \,, \tag{2.23}$$

where P_A is the cost of availability. Note that for the process to be potentially profitable we must have

$$P_W > P_A \tag{2.24}$$

because one unit of availabiity can give rise to at most one unit of work output. Using Eqs. (2.21)–(2.23) we obtain

$$\Pi = P_W \cdot W/\tau - P_A \cdot A/\tau \,, \tag{2.25}$$

$$\Pi = (P_W - P_A \eta_1) \cdot Q_1/\tau + (P_W - P_A \eta_2) \cdot Q_2/\tau \,, \tag{2.26}$$

and

$$\partial \Pi / \partial Q_i = (P_W - P_A \eta_i)/\tau > 0 \,. \tag{2.27}$$

Therefore, by the argument used in cases A–E above, we see that maximum profit occurs for maximum Q_1 and maximum Q_2. We note in passing that the same conclusion is obtained for a more general choice for the cost functions

$$C = P_A \cdot A/\tau + \alpha/\tau + \beta \,, \tag{2.23a}$$

where the constants α and β correspond to additional costs (e.g., capital investment costs, labor, maintenance, and depreciation), both those which are proportional to the rate of operation $1/\tau$ and those which are fixed.

III. ONE BRANCH OPTIMIZATION

Consider heat exchange branch i, $i = 1$ or 2, of our optimal cycle. By the assumptions of our model it is possible to instantaneously control the temperature of the working fluid by means of reversible work exchange with a work reservoir. We seek the function $T(t)$, i.e., temperature of the working fluid as a function of time, which corresponds to the optimal branch—the one yielding the largest heat exchange Q_i for the given entropy change and branch time τ_i. By integration in axiom (3), we find our objective function

$$Q_i = \int_0^{\tau_i} \kappa[T_i^{ex} - T(t)]\,dt = \kappa \int_0^{\tau_i} [T_i^{ex} - T(t)]\,dt \tag{3.1}$$

(κ is assumed to be constant independent of time or temperature). The constraint equation is

$$\int_0^{\tau_i} \frac{dQ_i}{T} = \int_0^{\tau_i} \frac{\kappa[T_i^{ex} - T(t)]}{T(t)}\,dt = \sigma_i = \text{const.} \tag{3.2}$$

This gives the modified Lagrangian

$$L = \kappa[T_i^{ex} - T(t)] - \lambda\{\kappa[T_i^{ex}/T(t) - 1]\} \,, \tag{3.3}$$

and the optimal $T(t)$ is obtained from the Euler–Lagrange equation

$$\partial L/\partial T = 0 = -\kappa + \lambda\kappa T_i^{ex}/T(t)^2 \,. \tag{3.4}$$

Thus, we get

$$T(t) = \sqrt{\lambda T_i^{ex}} = \text{constant} \equiv T_i \,. \tag{3.5}$$

The Legendre condition[11] results in

$$\frac{\partial^2 L}{\partial T^2} = -\frac{\lambda\kappa T_i^{ex}}{T^3} < 0 \tag{3.6}$$

since from constraint equation (3.2),

$$\lambda = \frac{T_i^{ex}}{(1 + \sigma/K\tau)^2} \,.$$

Thus, our solution corresponds to a maximum. Note that the free end point conditions[11]

$$\frac{\partial L}{\partial T}\bigg|_{t=0} = \frac{\partial L}{\partial T}\bigg|_{t=\tau_i} = 0$$

are automatically satisfied.

Equation (3.5) implies that the optimal way to induce an entropy change σ_i during time τ_i in a system in contact with a heat reservoir of constant temperature T_i^{ex} is by maintaining a constant temperature $T(t) = T_i$ in the system. T_i is obtained from Eq. (3.2):

$$T_i = T_i^{ex} \bigg/ \left(\frac{\sigma_i}{\kappa\tau_i} + 1\right) \,. \tag{3.7}$$

If the branch is defined by initial and final temperatures different from T_i, initial and final (instantaneous and reversible) adiabatic jumps are needed in order to bring the systems to the required internal temperature.

The results (3.4) and (3.7) obtained under the assumption of constant κ do not exclude the possibility that an isothermal branch may be interrupted by more adiabatic jumps in addition to those required at the beginning and at the end. Although the Euler equation (3.4) implies that transitions between adiabats occur on internal isotherms, it does not tell us how many different isotherms we require. We now show that we do best by using only a single isotherm along a given heat exchange branch.

Suppose that our optimal heat exchange branch i of time τ_i and entropy change σ_i consists of n different isotherms at temperatures T_j, $j = 1, \ldots, n$ (see Fig. 1). The total heat exchange is

$$Q_i = \sum_{j=1}^{n} \int_{t_{i-1}}^{t_j} dQ = \sum_{j=1}^{n} \int_{t_{i-1}}^{t_j} \kappa(T_i^{ex} - T_j)\,dt \tag{3.8}$$

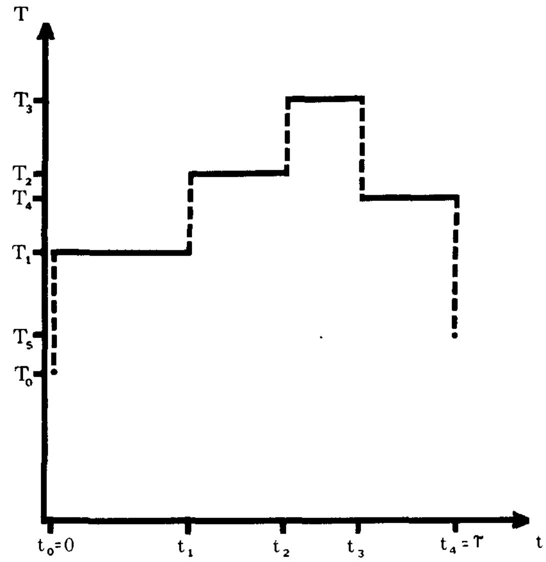

FIG. 1. Hypothetical $T(t)$ consistent with the Euler equation [Eq. (3.5)] using five adiabatic jumps and $n=4$ isotherms at temperatures T_j, $j=1, \cdots, n$.

$$= \sum_{j=1}^{n} \kappa (t_j - t_{j-1})(T_i^{ex} - T_j)$$

$$= \kappa \tau_i T_i^{ex} - \kappa \sum_{j=1}^{n} \Delta t_j T_j \,, \tag{3.9}$$

where $\Delta t_j = t_j - t_{j-1}$. If we can replace this branch with another $T(t)$ which has a larger Q_i and runs between the same adiabats, we improve the operation of the cycle. Maximizing Q_i with the constraint

$$\sigma_i = \sum_{j=1}^{n} \kappa \Delta t_j \frac{T_i^{ex}}{T_j} - \kappa \tau_i \tag{3.10}$$

and all t_j fixed is equivalent to minimizing $\sum_{j=1}^n \Delta t_j T_j$ with $\sum_{j=1}^n \Delta t_j / T_j$ constrained. This gives the modified objective function

$$L(T_1, \ldots, T_n) = \sum_{j=1}^{n} \Delta t_j \left(T_j + \frac{\lambda}{T_i} \right) \,. \tag{3.11}$$

The optimization conditions are

$$\frac{\partial L}{\partial T_j} = 0 = \Delta t_j \left(1 - \frac{\lambda}{T_j^2} \right) \,. \tag{3.12}$$

Thus,

$$T_j = \sqrt{\lambda} \,, \tag{3.13}$$

which implies that all the isotherms are at the same temperature T_i. This concludes our proof that the optimal process proceeds isothermally with initial and final adiabats needed to achieve the working temperature. (This last part of our derivation is called the staging or switching problem[11,18,19]: Solutions to optimal control problems normally have to be pieced together from solutions of the Euler equation and arcs which lie at the boundary of the controllability region. For Newton's law thermodynamics, such boundary arcs are instantaneous adiabatic processes.) The conclusion from the arguments above is *Theorem* 1: *Optimal operation of a finite time Carnot cycle* (*see Sec. I*) *in any of the senses A–F above requires that the cycle be of the Curzon–Ahlborn type,*[4,6] *i.e., that heat exchange branches take place with the temperature of the working fluid kept constant.*

If we add an inequality constraint bounding the rate at which work exchange can take place, the boundary arcs become maximum speed work exchange processes. Rubin[4] discusses the problem of maximum power with constraints bounding the rate at which work exchange can take place. He finds that adiabatic processes never occur, i.e., it always pays to be in contact with one of the reservoirs. Thus, Rubin's optimal cycles include heat exchange branches which take place with the working fluid not at constant temperature, in apparent contradiction to our theorem. These branches involve maximum rate work exchange and appear as the boundary arcs of Rubin's more constrained problem. Since our formalism does not constrain the possible rate of work exchange, we never encounter such heat exchange branches.

The constant temperature along a heat exchange branch is given by Eq. (3.7). For the Carnot type engine considered here $\sigma_1 + \sigma_2 = 0$. Defining $\sigma = \sigma_1$ we get

$$T_1 = \frac{T_1^{ex}}{1 + \frac{\sigma}{\kappa \tau_1}} \tag{3.14}$$

and

$$T_2 = \frac{T_2^{ex}}{1 - \frac{\sigma}{\kappa \tau_2}} \,. \tag{3.15}$$

The heats Q_1 and Q_2 are

$$Q_1 = \sigma T_1 = \frac{\sigma T_1^{ex}}{1 + \frac{\sigma}{\kappa \tau_1}} \,, \tag{3.16}$$

$$Q_2 = -\sigma T_2 = \frac{-\sigma T_2^{ex}}{1 - \frac{\sigma}{\kappa \tau_2}} \,. \tag{3.17}$$

Finally, we note that with $T(t) = T_i$ kept constant on each branch, the total entropy production rate dS_u/dt is

$$\frac{dS_u}{dt} = \frac{dQ}{dt} / T^{ex} - \frac{dQ}{dt} / T$$

$$= \kappa (T - T^{ex}) \left(\frac{1}{T^{ex}} - \frac{1}{T} \right) = \text{const.} \tag{3.18}$$

Thus, the total entropy production rate is constant along each branch. A similar conclusion was reached[1] for the more general class of heat engines considered in Ref. 1, where the total entropy production was chosen as the objective function.

IV. TIME ALLOCATION TO ADIABATS

We now show that our engine operates best in any of the senses A–F when the adiabatic branches proceed in zero time. By axiom (4) (Sec. I) we could make these branches run at any rate we choose, including infinitely fast. We now consider variations which reallocated some of the time spent on an adiabatic branch to a heat exchange branch, without altering the other two branches. Since our objective functions A–F depend only on Q_1, Q_2, and τ, such variations again alter the objective functions only by a change in Q_1. For concreteness, we consider a variation which reallocates time from the adiabatic branch 3 to the heat exchange branch 1 (see Fig. 2). Such variations alter τ_1 and τ_3 to $\tau_1+\epsilon$ and $\tau_3-\epsilon$, respectively, while keeping τ, Q_2, τ_2, and τ_4 constant. Further, we see from the fact that the heat exchange branch 2 is unaltered that the cycle has the same σ. Therefore, the heat exchange branch 1 must still run between the same adiabats, although somewhat slower. From Eq. (3.16) we see that

$$\frac{\partial Q_1}{\partial \tau_1} = \frac{\kappa T_1^{ex}}{\left(\frac{\kappa \tau_1}{\sigma}+1\right)^2} > 0 \,, \tag{4.1}$$

and so Q_1 is increased by an increase in τ_1. The same argument also applies for any reallocation of cycle time from an adiabatic branch to the heat exchange branch 2, since from Eq. (3.17)

$$\frac{\partial Q_2}{\partial \tau_2} = \frac{\kappa T_2^{ex}}{\left(\frac{\kappa \tau_2}{\sigma}-1\right)^2} > 0 \,. \tag{4.2}$$

Thus, reallocating time from an adiabatic branch to a heat exchange branch always improves the operation of the engine. We have thus proved *Theorem* 2: *Optimal operation of a finite time Carnot cycle in any of the senses A–F above requires that the adiabatic branches run in zero time.*

It is important to note that if we drop axiom (4) and impose constraints which do not allow zero time adiabats, our argument above still assures us that the optimal cycles have adiabatic branches which proceed at the maximal rates consistent with the constraints.

V. THE SPACE OF OPTIMAL FINITE TIME CARNOT CYCLES

Consider the set of optimal operations of a finite time Carnot cycle with given T_1^{ex}, T_2^{ex}, and κ. By Theorems 1 and 2, we know that such cycles have the following properties:

(I) Heat exchange branches take place with the temperature of the working fluid constant.

(II) Adiabatic branches take place in zero time.

It is therefore convenient to describe the set of optimal operations as a subset of the set of operations having properties I and II. We first introduce the following: *Definition: Two operations of a finite time Carnot cycle are equivalent if all the objective functions A–F are equal for the two operations.*

Recall from Sec. II that all our objective functions can be given in terms of Q_1, Q_2, and τ. We are now ready to prove the following lemma: *Lemma* 2: *Up to equivalence,*[20] *the set of all finite time Carnot cycles with properties I and II is a three dimensional space* $\bar{C}$, *with the global coordinate system* (τ_1, τ_2, σ). We will refer to $\bar{C}$ as the *Carnot space*.

We can see immediately that this space is three dimensional, since the three numbers Q_1, Q_2, and τ specify a process up to equivalence. We could show the second

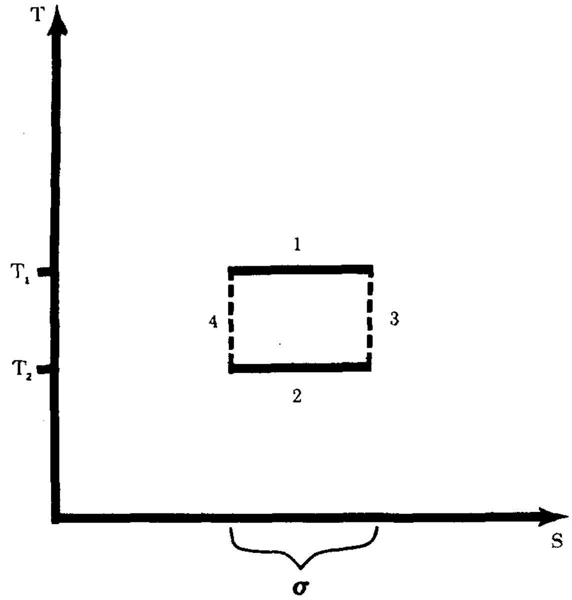

FIG. 2. (T, S) diagram of the path of the working fluid in its state space. Movement along the horizontal isotherms occurs at constant rates. Movement along the vertical adiabats occurs instantaneously.

part of the lemma by computing the Jacobian determinant

$$\left| \frac{\partial(\tau, Q_1, Q_2)}{\partial(\tau_1, \tau_2, \sigma)} \right| \tag{5.1}$$

of the transformation

$$(\tau_1, \tau_2, \sigma) \to (\tau, Q_1, Q_2) \,. \tag{5.2}$$

Because this direct approach soon gets bogged down with technical difficulties, we take an alternate route to showing that (τ_1, τ_2, σ) define a global system of coordinates. Our alternate route also shows us which cycles are equivalent.

First, consider a working fluid with only two degrees of freedom. We prove that τ_1, τ_2, T_1, T_2, σ, and S_0 determine the operation of the cycle, where S_0 is the smallest value of the entropy of the working fluid during the cycle (see Fig. 2). Indeed, from Eq. (3.5) we have

$$T(t) = \begin{cases} T_1\,, & 0<t<\tau_1\,, \\ T_2\,, & \tau_1<t<\tau_1+\tau_2\,, \end{cases} \tag{5.3}$$

$$S(t) = \begin{cases} S_0+\kappa\left(\frac{T_1^{ex}}{T_1}-1\right)t\,, & 0<t<\tau_1\,, \\ S_0+\sigma-\kappa\left(1-\frac{T_2^{ex}}{T_2}\right)(t-\tau_1)\,, & \tau_1<t<\tau_1+\tau_2\,. \end{cases} \tag{5.4}$$

Since the working fluid has only two degrees of freedom, we have determined the time evolution of the working fluid around the cycle. This, of course, also determines Q_1 and Q_2 [Eqs. (3.16) and (3.17), respectively], and hence the objective functions A–F. Furthermore, T_1 and T_2 can be determined from Eqs. (3.14) and (3.15),

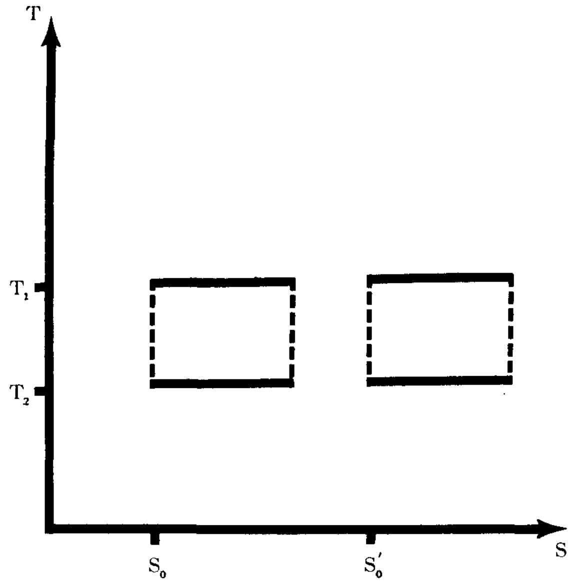

FIG. 3. Two operations are equivalent if and only if the (T, S) projections of their time trajectories are horizontal translates of each other.

leaving us with the four parameters τ_1, τ_2, σ, and S_0. From the equations (3.16), (3.17), and

$$\tau_1 + \tau_2 = \tau , \tag{5.5}$$

we can see that operations differing only by the value of S_0 are equivalent. Thus, up to equivalence we remain with the three parameters τ_1, τ_2, and σ.

Now consider a working fluid with n degrees of freedom. This means that the working fluid can be coupled to $n-1$ types of work reservoirs affecting changes in the extensive variables X_i with conjugate intensive variables Y_i, $i=1, \ldots, n-1$. The first law of thermodynamics has the form

$$dE = T\,dS + \sum_{i=1}^{n-1} Y_i\,dX_i \ . \tag{5.6}$$

Examples of such extensive–intensive variable pairs are provided by $X_1 = V$ (volume), $Y_1 = -p$ (pressure), and by $X_2 = H$ (magnetic field) and $Y_2 = M$ (magnetization). For more than one force, there is an infinite number of equivalent operations giving the time evolution $[T(t), S(t)]$ specified by Eqs. (5.3) and (5.4). Such operations differ from each other by trading off work input through force X_i for work input through force X_j. Since in our model either work input is reversible, such different operations are judged equivalent by any of our objective functions. We deduce that, generally, (τ_1, τ_2, σ) is a global coordinate system for the Carnot space $\overline{C}$.

The above argument started with information which specified the (T, S) indicator diagram of the cycle, using Eqs. (5.3) and (5.4) and the given values of τ_1, τ_2, and σ. Equivalent cycles with different S_0 have indicator diagrams which are horizontal translates of each other (see Fig. 3). Equivalent cycles which differed in the work input through forces X_i and X_j have the same projected indicator diagram $[T(t), S(t)]$ described by Eqs. (5.3) and (5.4). We have thus also proved *Lemma* 3:

Two finite time Carnot cycles are equivalent if and only if their projected indicator diagrams $[T(t), S(t)]$ *are horizontal translates of each other, i.e., if* $[T(t), S(t)] = [T'(t), S'(t) - S_0]$ *for some constant* S_0 *and for all* t.

Next we consider the boundaries of Carnot space. The coordinates τ_1, τ_2, and σ cannot assume arbitrary values. First of all, they must all be positive.[21] We further demand that the work produced by the engine

$$W = \sigma(T_1 - T_2) \tag{5.7}$$

be nonnegative. This requires $T_1 \geq T_2$, i.e.,

$$\frac{T_1^{\text{ex}}}{1 + \dfrac{\sigma}{\kappa\tau_1}} \geq \frac{T_2^{\text{ex}}}{1 - \dfrac{\sigma}{\kappa\tau_2}} \tag{5.8}$$

or

$$\frac{T_1^{\text{ex}}}{\tau_2} + \frac{T_2^{\text{ex}}}{\tau_1} \leq \frac{\kappa(T_1^{\text{ex}} - T_2^{\text{ex}})}{\sigma} \ . \tag{5.9}$$

The boundary of the permissible region [defined by the equality in Eq. (5.9)] is a cone in the (τ_1, τ_2, σ) space. For constant σ sections this boundary constitutes an hyperbola with vertical and horizontal asymptotes:

$$\tau_1 = \frac{\sigma T_2^{\text{ex}}}{\kappa(T_1^{\text{ex}} - T_2^{\text{ex}})} = \frac{\sigma(1 - \eta_{\text{rev}})}{\kappa\,\eta_{\text{rev}}} , \tag{5.10}$$

$$\tau_2 = \frac{\sigma T_1^{\text{ex}}}{\kappa(T_1^{\text{ex}} - T_2^{\text{ex}})} = \frac{\sigma}{\kappa\eta_{\text{rev}}} \ . \tag{5.11}$$

On this boundary the work W vanishes and we shall refer to it as the zero work cone or, for constant σ, as the work hyperbola. Several such hyperbolas are shown by Fig. 4 (solid curves).

The region of Carnot space accessible to a given physical apparatus is bounded by yet another surface $\sigma = \sigma_{\max}(\tau_1,$

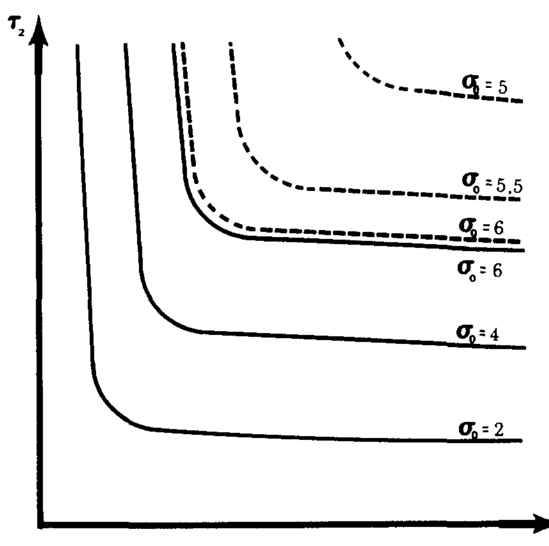

FIG. 4. Constant $\sigma (= \sigma_0)$ sections of the Carnot subspace $\overline{C}$ in the (τ_1, τ_2, σ) space. On these sections $\overline{C}$ is bounded by a $W = 0$ line (solid curves) and by a $\sigma_{\max} = \sigma_0$ line (dashed curves), defined by Eqs. (5.9) (with equality sign) and (5.12), respectively. The parameters used for the calculations were $\sigma_{\text{rev}} = 4$ e.u., $C_v = 3R/2$, $\kappa = 1$ e.u./sec, $T_1^{\text{ex}} = 600\,^\circ\text{K}$, $T_2^{\text{ex}} = 300\,^\circ\text{K}$.

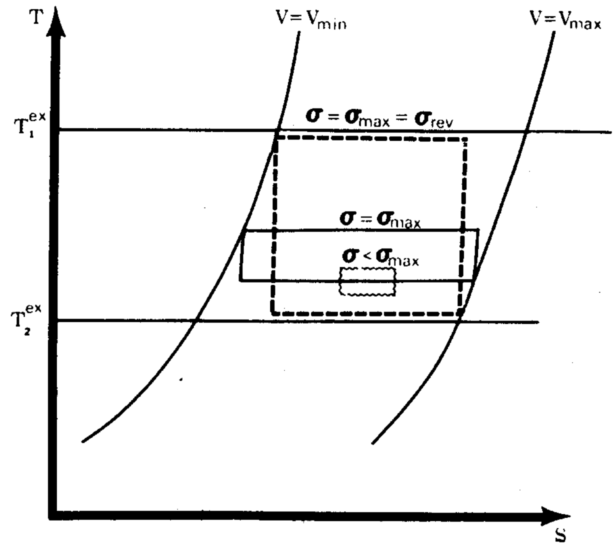

FIG. 5. The accessible region of (T, S) space of the working fluid specified by $V_{min} \le V \le V_{max}$ and $T_2^{ex} \le T \le T_1^{ex}$. The accessible region for the working fluid is bounded by the lines $T = T_1^{ex}$, $T = T_2^{ex}$, $T = T(S, V_{min})$, and $T = T(S, V_{max})$. The latter two lines are positively sloped, the slopes being $(\partial T/\partial S)_V = T/C_V$, calculated at V_{min} and V_{max}, respectively. (Note that for an ideal gas these two lines are parallel to each other.) One trajectory is shown with $\sigma = \sigma_{max}$, i.e., with the working fluid reaching V_{min} and V_{max}. For one of the other trajectories, $\sigma = \sigma_{rev}$, which is the smallest possible σ consistent with $\sigma = \sigma_{max}(\tau_1, \tau_2)$.

τ_2). This boundary is due to limits on the values of the mechanical variables, e.g., size or strength of the cylinder containing the working fluid. Consider, for example, 1 mole of an ideal gas in a cylinder equipped with a piston. Any real cylinder will have a maximum volume V_{max} and a minimum volume V_{min} or, alternatively, a maximum pressure p_{max} which the walls can withstand. In either case such limitations restrict the value that σ can attain. The largest value σ_{max} is different for different τ_1 and τ_2. For specified V_{max} and V_{min}, $\sigma_{max}(\tau_1, \tau_2)$ is obtained implicitly[1] as the solution σ of the equation

$$\sigma = \sigma_{rev} + C_V \ln\left[\left(1 + \frac{\sigma}{\kappa\tau_1}\right) \Big/ \left(1 - \frac{\sigma}{\kappa\tau_2}\right)\right] , \qquad (5.12)$$

where

$$\sigma_{rev} = R \ln(V_{max}/V_{min}) + C_V \ln(T_2^{ex}/T_1^{ex}) . \qquad (5.13)$$

For specified V_{max} and p_{max}, σ_{max} is obtained[1] as the solution of

$$\sigma = \sigma_{rev} + C_p \ln\left(1 + \frac{\sigma}{\kappa\tau_1}\right) - C_V \ln\left(1 - \frac{\sigma}{\kappa\tau_2}\right) , \qquad (5.14)$$

where now

$$\sigma_{rev} = R \ln(V_{max} p_{max}/R) + C_V \ln T_2^{ex} - C_p \ln T_1^{ex} . \qquad (5.15)$$

These equations follow from the ideal gas equation of state

$$S = S_0 + R \ln(V/V_0) + C_V \ln(T/T_0) \qquad (5.16)$$

and the requirement that the working fluid reaches the extreme values of V and p. Note that σ_{rev} is defined here as the entropy carried by the working fluid in a reversible process which satisfies these requirements. Generally, the boundary surface $\sigma = \sigma_{max}$ will depend on the equations of state of the working fluid and on the mechanical limits of our machine. As we shall see below, the qualitative features of this surface are the same for most situations. In order to examine these features for the ideal gas example we refer to Fig. 5, which schematically illustrates the dependence of σ on τ_1 and τ_2. For large cycle time τ, $T_1 \sim T_1^{ex}$ and $T_2 \sim T_2^{ex}$ and $\sigma_{max}(\tau_1, \tau_2)$ approaches its smallest value σ_{rev} as τ_1 and τ_2 approach infinity. For faster operation, σ_{max} increases slowly, reaching a limiting value at the fastest operation. This dependence of σ_{max} on τ is graphed in Fig. 6 for $\tau_1 = \tau_2$. Figure 6 also displays the $W=0$ boundary curve, which for $\tau_1 = \tau_2 = \frac{1}{2}\tau$ is a straight line in the $\sigma - \tau$ plane with a slope $\kappa(T_1^{ex} - T_2^{ex})/(T_1^{ex} + T_2^{ex}) > 0$ [cf. the equality in Eq. (5.9)].

The qualitative features of the $\sigma_{max}(\tau)$ dependence apply quite generally. Consider a working fluid characterized by n degrees of freedom and subject to $2(n-1)$ mechanical limitations. These limitations restrict the accessible states of the working fluid to a boxlike region very similar to the one pictured in Fig. 5. Two edges will be along the line $T = T_1^{ex}$, while the other two edges will be positively sloped. For the example of a gas in a cylinder restricted by maximum volume and maximum pressure conditions, these slopes are

$$(\partial S/\partial T)_p = C_p/T > 0 , \qquad (5.17)$$

$$(\partial S/\partial T)_V = C_V/T > 0 . \qquad (5.18)$$

Generally, these slopes are C_X/T for some mechanical parameters X which are held at their extreme values. Provided such heat capacities are positive and of moderate size, the qualitative features of the $\sigma = \sigma_{max}(\tau_1, \tau_2)$ surface will be the same as for the ideal gas in a pressure and volume constrained cylinder. If the C_X's can be considered constant in the range of interest, we get the following equation for σ_{max}:

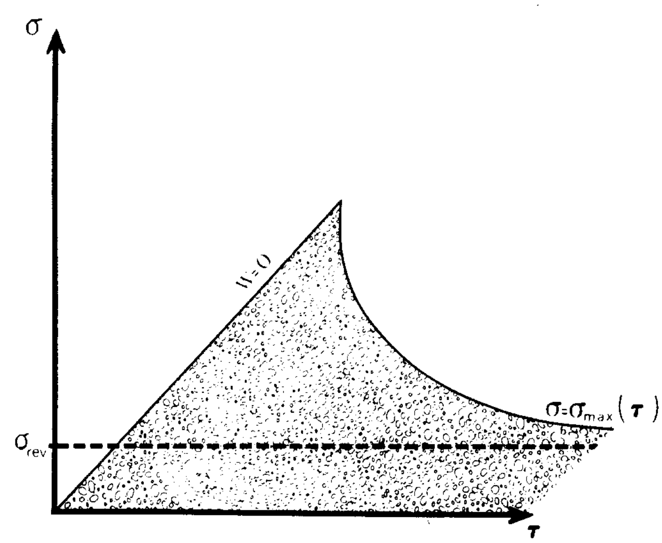

FIG. 6. A cross section of Carnot space (shaded region) by the plane $\tau_1 = \tau_2$. $\sigma_{max}(\tau)$ is $\sigma_{max}(\tau_1, \tau_2)$ with $\tau_1 = \tau_2 = \tau/2$. For an ideal gas working fluid, the value of σ at the peak where the $\sigma = \sigma_{max}(\tau)$ and the $W=0$ boundaried intersect is $\sigma = R\ln(V_{max}/V_{min})$.

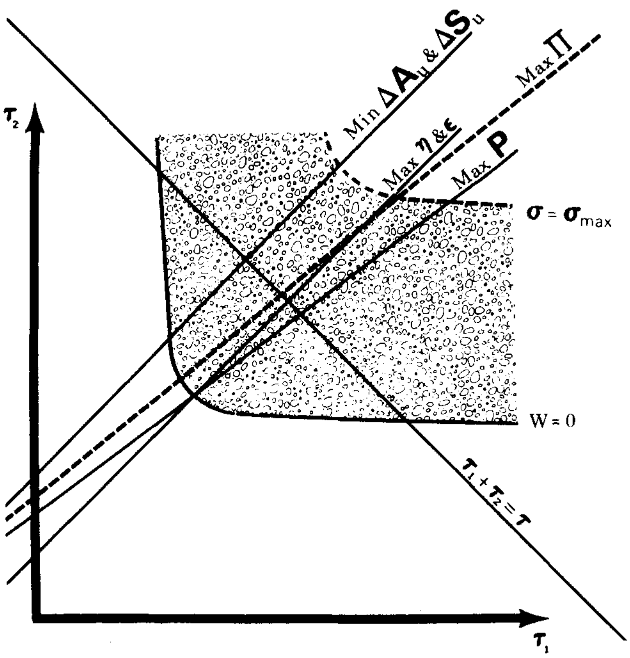

FIG. 7. A constant σ section in the (τ_1, τ_2, σ) space displaying the boundaries $W=0$ and $\sigma_{max}(\tau_1, \tau_2) = \sigma$ of Carnot space and the lines for maximum power, maximum efficiency or effectiveness, maximum profit and minimum entropy, or availability changes [extrema obtained for fixed σ and τ with respect to the time allocation (τ_1, τ_2)]. The points of optimal operation lie at the intersection of the line $\tau_1 + \tau_2 = \tau$ with the corresponding optimum line.

$$\sigma_{max} = \sigma_{rev} + C_X \ln\left(1 + \frac{\sigma_{max}}{\kappa\tau_1}\right) - C_{X'} \ln\left(1 - \frac{\sigma_{max}}{\kappa\tau_2}\right) . \tag{5.19}$$

If $C_X = C_{X'}$, the constant $\sigma = \sigma_0$ slices of the $\sigma = \sigma_{max}(\tau_1, \tau_2)$ surface are the hyperbolas

$$\frac{\kappa}{\sigma_0}[B(\sigma_0) - 1] = \frac{1}{\tau_1} + \frac{B(\sigma_0)}{\tau_2} , \tag{5.20}$$

where

$$B(\sigma) = \exp[(\sigma - \sigma_{rev})/C_X] . \tag{5.21}$$

These hyperbolas appear as dashed curves in Fig. 4. For any choice of σ the working region is bounded in principle by one solid and one dashed line in Fig. 4. It should be noted, however, that the equation $\sigma_0 = \sigma_{rev} + C_X \ln(1 + \sigma_0/\kappa\tau_1) - C_{X'} \ln(1 - \sigma_0/\kappa\tau_2)$ viewed as an equation for a curve in the $\tau_1, \tau_2 \geq 0$ quadrant of the (τ_1, τ_2) plane has a solution only for $\sigma_0 > \sigma_{rev}$. For $\sigma_0 \leq \sigma_{rev}$ only the $W=0$ boundary appears in Fig. 4. For $\sigma_0 > \sigma_{rev}$ the $\sigma \leq \sigma_{max}(\tau_1, \tau_2)$ condition also becomes important and the section of $\overline{C}$ in the $\sigma = \sigma_0$ plane is the crescent shape region lying between the hyperbolas (5.9) and (5.20) (the latter hyperbola being an approximation as discussed above). This region is seen in Figs. 7 and 8.

Finally, we also require that $\sigma \geq 0$. Thus, $\overline{C}$ is the region in (τ_1, τ_2, σ) space which is bounded by the surfaces $W=0$, $\sigma = \sigma_{max}$, and $\sigma = 0$. A vertical (perpendicular to the constant σ plane) cross section of $\overline{C}$ by the plane $\tau_1 = \tau_2$ is shown in Fig. 6. Horizontal sections of $\overline{C}$ with planes of constant $\sigma = \sigma_0$ are shown in Figs. 4, 7, and 8, as described above.

VI. CARNOT SPACE OPTIMIZATIONS

We now find the optimal operations of our Carnot cycle for each of the objective functions A–F. Since the optimal values of these objective functions are to be used as criteria of merit for judging the operation of engines with measured values of σ and τ, we first optimize over the time allocation τ_1, τ_2 with constrained values of σ and τ. Optimizations with respect to σ and τ are left to Sec. VII. In general, if one optimizes a function subject to constraints and later optimizes over the values of the constraints, one ends up with the absolute optimum of the function. Consider the objective function $F(\tau_1, \tau_2, \sigma)$. To optimize with given σ and τ, we hold σ constant and add the constraint of given τ by the use of a Lagrange multiplier λ. Then

$$\left[\frac{\partial}{\partial\tau_1}(F - \lambda\tau)\right]_{\tau_2,\sigma} = \left[\frac{\partial}{\partial\tau_2}(F - \lambda\tau)\right]_{\tau_1,\sigma} = 0 \tag{6.1}$$

holds at any extremum. On elimination of λ, this assumes the convenient form

$$(\partial F/\partial\tau_1)_{\tau_2,\sigma} = (\partial F/\partial\tau_2)_{\tau_1,\sigma} . \tag{6.2}$$

For the unconstrained optimizations of Sec. VII, we will have the equations

$$(\partial F/\partial\tau_1)_{\tau_2,\sigma} = (\partial F/\partial\tau_2)_{\tau_1,\sigma} = (\partial F/\partial\sigma)_{\tau_1,\tau_2} = 0 . \tag{6.3}$$

Note that Eq. (6.3) is a special case of Eq. (6.2).

Since all our F's can be expressed in terms of Q_1, Q_2, and τ, we will have frequency occasion to refer to the derivatives

$$(\partial Q_1/\partial\tau_1) = \kappa T_1^{ex}/[(\kappa\tau_1/\sigma) + 1]^2 , \tag{6.4}$$

$$(\partial Q_2/\partial\tau_2) = \kappa T_2^{ex}/[(\kappa\tau_2/\sigma) - 1]^2 , \tag{6.5}$$

$$(\partial Q_1/\partial\sigma) = T_1^{ex}/[(\sigma/\kappa\tau_1) + 1]^2 , \tag{6.6}$$

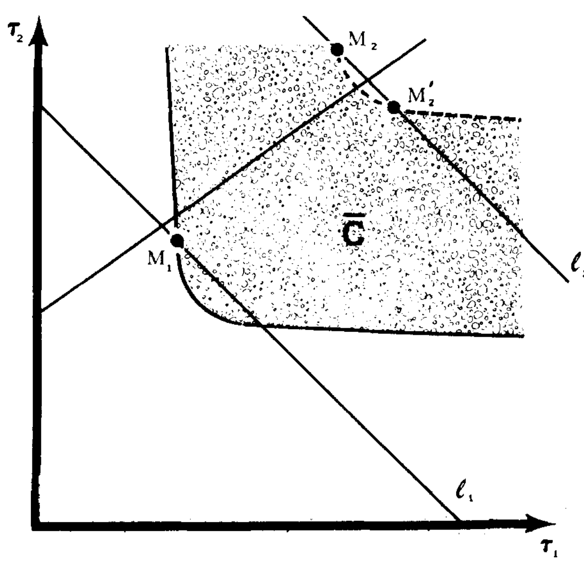

FIG. 8. The $\sigma = \sigma_0$ slice of Carnot space is the shaded region bounded by the hyperbolas (5.9) and (5.20). The positively sloped line is the line of optimal Ψ, where Ψ can be any of the objective functions A–F. l_1 and l_2 are lines of given $\tau_0 = \tau_1 + \tau_2$. The values of τ_0 on lines l_1 and l_2 were chosen so that their points of intersection with the line of optimal Ψ fall outside $\overline{C}$.

$$(\partial Q_2/\partial\sigma) = T_2^{ex}/[(\sigma/\kappa\tau_2) - 1]^2 , \quad (6.7)$$

obtained from Eqs. (3.16) and (3.17). Note also that

$$\partial Q_1/\partial\tau_2 = \partial Q_2/\partial\tau_1 = 0 . \quad (6.8)$$

A. Case A: Maximum Power

Using the objective functions $F = W/\tau$, Eq. (6.2) for constant τ gives

$$\partial W/\partial\tau_2 = \partial W/\partial\tau_2 . \quad (6.9)$$

This reduces to

$$\partial Q_1/\partial\tau_1 = \partial Q_2/\partial\tau_2 , \quad (6.10)$$

which gives

$$\tau_2 = \sqrt{\frac{T_2^{ex}}{T_1^{ex}}}\,\tau_1 + \left(1 + \sqrt{\frac{T_2^{ex}}{T_1^{ex}}}\right)\frac{\sigma}{\kappa} . \quad (6.11)$$

For various choices of σ and τ, Eq. (6.11) represents a plane in (τ_1, τ_2, σ). The intersection of this plane with a plane of constant σ is the maximum power line of slope $\sqrt{T_2^{ex}/T_2^{ex}}<1$ shown in Fig. 7. Solving Eq. (6.11) simultaneously with

$$\tau_1 + \tau_2 = \tau \quad (6.12)$$

gives

$$\tau_1 = \frac{\tau}{[1 + (T_2^{ex}/T_1^{ex})^{1/2}]} - \frac{\sigma}{\kappa} \quad (6.13)$$

and

$$\tau_2 = \frac{\tau}{[1 + (T_1^{ex}/T_2^{ex})^{1/2}]} + \frac{\sigma}{\kappa} . \quad (6.14)$$

These give [using $W = Q_1 + Q_2$ and Eqs. (3.16) and (3.17)]

$$P = W/\tau = (T_1^{ex} - T_2^{ex})(\sigma/\tau) - (\sqrt{T_1^{ex}} + \sqrt{T_2^{ex}})^2\frac{\sigma^2}{\kappa\tau^2} . \quad (6.15)$$

Note that the resulting optimal power depends only on the ratio σ/τ rather than on σ and τ themselves.

B. Cases B and C: Maximum efficiency and effectiveness

As shown in Sec. II. C, these two objective functions are equivalent, being maximum for maximum Q_2/Q_1. Using

$$\frac{\partial(Q_2/Q_1)}{\partial\tau_1} = \frac{T_2^{ex}}{T_1^{ex}}\,\frac{1}{1 - \frac{\sigma}{\kappa\tau_2}}\,\frac{\sigma}{\kappa\tau_1^2} , \quad (6.16)$$

$$\frac{\partial(Q_2/Q_1)}{\partial\tau_2} = \frac{T_2^{ex}}{T_1^{ex}}\,\frac{1 + \sigma/\kappa\tau_1}{(1 - \sigma/\kappa\tau_2)^2}\,\frac{\sigma}{\kappa\tau_2^2} , \quad (6.17)$$

and (for the fixed σ, fixed τ optimization)

$$\frac{\partial(Q_2/Q_1)}{\partial\tau_1} = \frac{\partial(Q_2/Q_1)}{\partial\tau_2} \quad (6.18)$$

gives

$$\tau_2 = \tau_1 + \sigma/\kappa , \quad (6.19)$$

which again represents a plane in the (τ_1, τ_2, σ) space for various choices of σ and τ. For constant σ, Eq. (6.19) is the maximum efficiency (η) and effectiveness (ϵ) line of slope 1 and intercept σ/κ shown in Fig. 7. Solving with $\tau_1 + \tau_2 = \tau$ these equations give

$$\tau_1 = \tau/2 - \sigma/2\kappa , \quad (6.20)$$

$$\tau_2 = \tau/2 + \sigma/2\kappa , \quad (6.21)$$

and on substitution

$$\eta = 1 - \frac{T_2^{ex}}{T_1^{ex}}\left(\frac{1 + \frac{\sigma}{\kappa\tau}}{1 - \frac{\sigma}{\kappa\tau}}\right)^2 \quad (6.22)$$

and

$$\epsilon = \frac{T_1^{ex}\left(1 - \frac{\sigma}{\kappa\tau}\right)^2 - T_2^{ex}\left(1 + \frac{\sigma}{\kappa\tau}\right)^2}{(T_1^{ex} - T_{en})\left(1 - \frac{\sigma}{\kappa\tau}\right)^2 + (T_2^{ex} - T_{en})\left(1 + \frac{\sigma}{\kappa\tau}\right)^2} . \quad (6.23)$$

Note that η and ϵ depend only on σ/τ.

C. Cases D and E: Minimum entropy production and loss of availability

These two objective functions are again equivalent. We work with

$$\Delta S_u = -(Q_1/T_1^{ex} + Q_2/T_2^{ex}) . \quad (6.24)$$

The constrained σ, τ condition (6.2) becomes

$$\frac{1}{T_1^{ex}}\,\frac{\partial Q_1}{\partial\tau_1} = \frac{1}{T_2^{ex}}\,\frac{\partial Q_2}{\partial\tau_2} , \quad (6.25)$$

which gives

$$\tau_2 = \tau_1 + 2\sigma/\kappa . \quad (6.26)$$

Again, for various choices of σ and τ, this is a plane whose intersection with a plane of constant σ is the minimum ΔS_u and ΔA line of slope 1 and intercept $2\sigma/\kappa$ shown in Fig. 7. Substituting, we get $\tau_1 = \tau/2 - 2\sigma/\kappa$, $\tau_2 = \tau/2 + 2\sigma/\kappa$; hence [using Eqs. (6.24), (3.16), (3.17), and (2.20)]

$$\Delta S_u = \frac{4\sigma^2}{\kappa\tau} \quad (6.27)$$

and

$$\Delta A_u = T_{en}\frac{4\sigma^2}{\kappa\tau} . \quad (6.28)$$

Note that while these quantities depend on the values of σ and τ, the corresponding rates $\Delta S_u/\tau$ and $\Delta A_u/\tau$ depend on σ/τ.

D. Case F: Maximum profit

As shown by Eq. (2.25),

$$\Pi = (P_W - P_A\eta_1)(Q_1/\tau) + (P_W - P_A\eta_2)(Q_2/\tau) , \quad (6.29)$$

which in terms of τ_1, τ_2, and σ becomes

$$\Pi = P_W(\sigma/\tau)\left[\frac{(1 - \eta_1 P_A/P_W)T_1^{ex}}{1 + \sigma/\kappa\tau_1} - \frac{(1 - \eta_2 P_A/P_W)T_2^{ex}}{1 - \sigma/\kappa\tau_2}\right] . \quad (6.30)$$

Introducing the definitions

$$T_1^{\rm eff} = T_1^{\rm ex}\left(1 - \frac{P_A}{P_W}\,\eta_1\right) , \qquad (6.31)$$

$$T_2^{\rm eff} = T_2^{\rm ex}\left(1 - \frac{P_A}{P_W}\,\eta_2\right) , \qquad (6.32)$$

and

$$Q_1^{\rm eff} = \frac{\sigma T_1^{\rm eff}}{(1+\sigma/\kappa\tau_1)} , \qquad (6.33)$$

$$Q_2^{\rm eff} = \frac{-\sigma T_2^{\rm eff}}{(1-\sigma/\kappa\tau_2)} \qquad (6.34)$$

[in analogy with Eqs. (3.16) and (3.17)], and $Q_1^{\rm eff}$ and $Q_2^{\rm eff}$ are the heats which would pass from the working fluid undergoing the cycle specified by τ_1, τ_2, and σ if the reservoirs were at the temperatures $T_1^{\rm eff}$ and $T_2^{\rm eff}$], we get

$$\Pi = P_W(Q_1^{\rm eff} + Q_2^{\rm eff})/\tau . \qquad (6.35)$$

We see from Eqs. (6.35) and (2.3) that Π depends on the τ_i exactly as the power P. Thus, optimization with constrained σ and τ give

$$\tau_2 = (T_1^{\rm eff}/T_2^{\rm eff})^{1/2}\tau_1 + [1 + (T_1^{\rm eff}/T_2^{\rm eff})^{1/2}]\,\sigma/\kappa . \qquad (6.36)$$

This is once again a plane for various σ and τ. For constant σ, this plane reduces to the maximum profit line of slope $T_1^{\rm eff}/T_2^{\rm eff}$ shown in Fig. 7.

The maximum profit line always lies between the maximum power line and the minimum entropy production line. It merges with the maximum power line when

$$T_1^{\rm eff}/T_2^{\rm eff} \to T_1^{\rm ex}/T_2^{\rm ex} , \qquad (6.37)$$

i.e., when

$$\left(1 - \frac{P_A}{P_W}\,\eta_1\right)\Big/\left(1 - \frac{P_A}{P_W}\,\eta_2\right) \to 1 . \qquad (6.38)$$

This occurs for $P_A/P_W \to 0$, i.e., $P_W \gg P_A$. *As the price of work becomes very large compared with the price of availability, profit maximization approaches power maximization.* On the other hand, as the price of work becomes comparable with the price of availability, $P_W \to P_A$:

$$\left(1 - \frac{P_A}{P_W}\,\eta_1\right)\Big/\left(1 - \frac{P_A}{P_W}\,\eta_2\right) \to (1-\eta_1)/(1-\eta_2) = \frac{T_2^{\rm ex}}{T_1^{\rm ex}} . \qquad (6.39)$$

Therefore,

$$T_1^{\rm eff}/T_2^{\rm eff} \to 1 \qquad (6.40)$$

and the maximum profit line [Eq. (6.36)] becomes

$$\tau_2 = \tau_1 + 2\sigma/\kappa , \qquad (6.41)$$

which is the minimum entropy production condition (6.26). Thus, *as the price of work approaches the price of availability, profit maximization approaches entropy production minimization, i.e., minimum waste of availability*. For any intermediate P_A/P_W, the maximum profit line lies between the maximum power line and the minimum entropy production line.

To obtain the maximum profit for the given values of σ and τ, we again solve Eq. (6.36) with

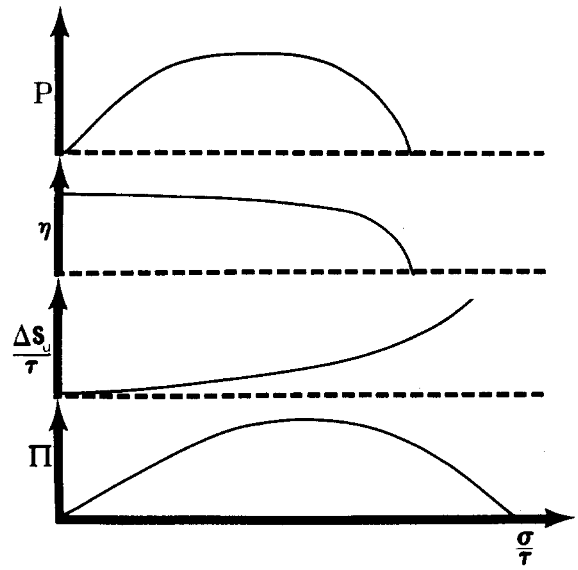

FIG. 9. Various objective functions optimized with respect to time allocation (τ_1, τ_2) for fixed τ and σ, plotted as functions of σ/τ.

$$\tau_1 + \tau_2 = \tau \qquad (6.42)$$

to give

$$\tau_1 = \tau/[1 + (T_2^{\rm eff}/T_1^{\rm eff})^{1/2}] - \sigma/\kappa , \qquad (6.43)$$

$$\tau_2 = \tau/[1 + (T_1^{\rm eff}/T_2^{\rm eff})^{1/2}] + \sigma/\kappa , \qquad (6.44)$$

and

$$\Pi = P_W\left[(T_1^{\rm eff} - T_2^{\rm eff})\,\frac{\sigma}{\tau} - (\sqrt{T_1^{\rm eff}} + \sqrt{T_2^{\rm eff}})^2\,\frac{\sigma^2}{\kappa\tau^2}\right] . \qquad (6.45)$$

Note that Π depends only on σ/τ.

The optimal functions P [Eq. (6.15)], η [Eq. (6.22)], $\Delta S_u/\tau$ [Eq. (6.27)], and Π [Eq. (6.45)] are schematically plotted in Fig. 9 as functions of the ratio σ/τ.

E. Boundary optima

Above we found the interior optima[22] for our objective functions A–F with constrained σ and τ. This involved finding the points (τ_1, τ_2, σ) on the line

$$l = \{(\tau_1, \tau_2, \sigma) ; \ \tau_1 + \tau_2 = \tau_0 \text{ and } \sigma = \sigma_0\} , \qquad (6.46)$$

which made the respective objective functions optimal. The results of these optimizations are summarized in Fig. 7. However, these optimizations were performed over the entire space (τ_1, τ_2, σ) without regard to the fact that we are interested only in optimal (τ_1, τ_2, σ) which lie in the three dimensional region $\overline{C}$ bounded by the three surfaces

$$\sigma = 0 , \qquad (6.47)$$

$$W = 0 , \qquad (6.48)$$

$$\sigma = \sigma_{\max}(\tau_1, \tau_2) . \qquad (6.49)$$

We now briefly describe what happens when the optimal point (τ_1, τ_2, σ) found by the methods above falls outside $\overline{C}$. Let Ψ stand for any of the objective functions

A–F. Then, as shown above, the point of optimal Ψ lies at the intersection of one of the lines (6.11), (6.19), (6.26), or (6.36) and the line l. Provided $\ln\bar{C} \neq 0$, such point of intersection can fall outside $\bar{C}$ in one of the two ways pictured in Fig. 8. For both these cases, optimal operation in $\bar{C}$ takes place at the nearest point of $\bar{C}$ on l, i.e., on the boundary. In the example considered (Fig. 8), when τ_0 is too small for the point of optimal operation to lie within $\bar{C}$ (line l_1 in Fig. 8), there is only one nearest boundary point on l (point M_1 in Fig. 8). When τ_0 is too large for the point of optimal operation to lie in $\bar{C}$ (line l_2 in Fig. 8), there are two nearest boundary points (M_2 and M_2' in Fig. 8). We forego the analysis of which boundary point (M_2 or M_2') is optimal in this case since such analysis is incidental to our development and since the results depend on the nature of the $\sigma=\sigma_{max}(\tau_1, \tau_2)$ surface, i.e., on the working fluid.

VII. OPTIMIZATIONS OVER σ AND τ

We now consider unconstrained optimizations in $\bar{C}$, i.e., we optimize with respect to the choices of σ and τ. As we saw in Sec. VI, our optimized objective functions P [case A, Eq. (6.15)], η [case B, Eq. (6.22)], ϵ [case C, Eq. (6.23)], and Π [case F, Eq. (6.45)] depend only on σ/τ, while the optimized objective functions ΔS_u [case D, Eq. (6.27)] and ΔA_u [case E, Eq. (6.28)] depend only on σ^2/τ. In neither case do the optimality conditions

$$\partial\Psi/\partial\sigma = \partial\Psi/\partial\tau = 0 ,$$

where Ψ represents any of the objective functions A–F, determine unique values of σ and τ. Instead, they determine values of σ/τ for Cases A, B, C, and F and σ^2/τ for cases D and E.[23] By optimizing with respect to σ/τ or σ^2/τ we obtain at once the results for (1) unconstrained optimizations, (2) optimizations over τ with given σ, and (3) optimizations over σ with given τ.

A. Case A: Maximum power

If we optimize the power

$$P=(T_1^{ex}-T_2^{ex})\frac{\sigma}{\tau}-(\sqrt{T_1^{ex}}+(\sqrt{T_2^{ex}})^2\frac{\sigma^2}{\kappa\tau^2} \tag{7.1}$$

with respect to σ/τ, we find

$$\frac{\sigma}{\tau}=\frac{\kappa}{2}\frac{\sqrt{T_1^{ex}}-\sqrt{T_2^{ex}}}{\sqrt{T_1^{ex}}+\sqrt{T_2^{ex}}} \tag{7.2}$$

and

$$P=\tfrac{1}{4}\kappa(\sqrt{T_1^{ex}}-\sqrt{T_2^{ex}})^2 . \tag{7.3}$$

Solving Eq. (7.2) simultaneously with Eqs. (6.13) and (6.14), we find that power assumes its absolute maximum on all points (τ_1, τ_2, σ) on the line

$$\frac{\tau}{2}=\tau_1=\tau_2=\frac{\sigma}{\kappa}\frac{\sqrt{T_1^{ex}}+\sqrt{T_2^{ex}}}{\sqrt{T_1^{ex}}-\sqrt{T_2^{ex}}} \tag{7.4}$$

shown in Fig. 10. The efficiency η of any point on this line is obtained from Eqs. (2.4) (3.16), (3.17), and (7.4) in the form

$$\eta=1-(T_2^{ex}/T_1^{ex})^{1/2} . \tag{7.5}$$

This result was first obtained by Curzon and Ahlborn.[6]

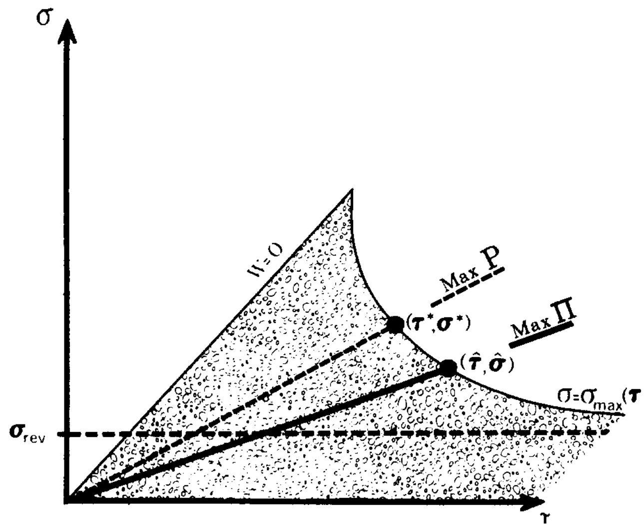

FIG. 10. $\tau_1=\tau_2\,(=\tau/2)$ section of Carnot space showing the lines of optimal unconstrained power and profit. (τ^*, σ^*) and $(\hat{\tau}, \hat{\sigma})$ are the points of intersection of these lines with the surface $\sigma=\sigma_{max}(\tau_1, \tau_2)$.

The result (7.2) which corresponds to the maximum P line on Fig. 10 holds for the maximum power operation of the engine only if σ and τ also satisfy $W\geq 0$ and $\sigma\leq\sigma_{max}$, i.e., if $(\tau_1, \tau_2, \sigma)\in\bar{C}$. $\bar{C}$ is bounded by the $W=0$ and $\sigma=\sigma_{max}(\tau_1, \tau_2)$ curves in Fig. 10. The choice of σ/τ which lies on this maximum power line leads to the unconstrained maximum (7.3) for P. Consider now the case where $\sigma=\sigma_0$ is given and a value for τ has to be found which maximizes P under this σ constraint. Let σ^* and τ^* denote the σ and τ values, respectively, at the intersection of the maximum P line with the $\sigma=\sigma_{max}$ boundary (Fig. 10). For an ideal gas working fluid in a cylinder with given V_{max} and V_{min} we obtain

$$\sigma^*=\sigma_{rev}+\tfrac{1}{2}C_V\ln(T_1^{ex}/T_2^{ex}) , \tag{7.6}$$
$$\tau^*=(2\sigma^*/\kappa)(\sqrt{T_1^{ex}}+\sqrt{T_2^{ex}})/(\sqrt{T_1^{ex}}-\sqrt{T_2^{ex}}) ,$$

where σ_{rev} is given by Eq. (5.13). As long as $\sigma_0<\sigma^*$, the maximum power operation of the engine is still determined from Eqs. (7.2) and (7.3). However, if $\sigma_0>\sigma^*$, the solution for τ from Eq. (7.2) (with $\sigma=\sigma_0$) is larger than τ^*. When this is the case we see from the shape of the $P(\sigma/\tau)$ curve in Fig. 9 that we should make τ as large as possible consistent with the relation (6.11) and the given value σ_0 of σ. For various $\sigma>\sigma^*$ the optimal point traces part of the curve of intersection of the surface $\sigma=\sigma_{max}$ and the optimal power plane (6.11). Optimization with constrained τ shows similar behavior. For $\tau<\tau^*$ we choose a σ which makes Eq. (7.2) hold, while for $\tau>\tau^*$ we choose the largest possible σ. This generates another portion of the curve of intersection of $\sigma=\sigma_{max}$ and Eq. (6.11).

B. Cases B, C, D, and E: Maximum efficiency, effectiveness, minimum entropy production, and loss of availability

These cases are conveniently lumped together. Efficiency and effectiveness are monotonic in σ/τ, and ΔS_u and ΔA_u are monotonic in σ^2/τ (see Fig. 9). Each of these objective functions takes on its absolute minimum for $\sigma/\tau = \sigma^2/\tau = 0$, i.e., on the bounding surfaces

$$\sigma = 0\ ; \quad \tau_1 > 0, \quad \tau_2 > 0 \tag{7.7}$$

and

$$\sigma_{rev} > \sigma > 0\ ; \quad \tau_1\ ,\ \tau_2 = +\infty\ . \tag{7.8}$$

Note that the constrained σ, τ optimality conditions (6.19) and (6.26) do not play a role here since the optimal values of these objective functions are achieved at all points on the surfaces (7.7) and (7.8). Optimization subject to a constrained cycle time τ requires that $\sigma = 0$. Optimization subject to a constrained σ requires τ to be as large as possible. For $\sigma < \sigma_{rev}$ this means $\tau = +\infty$, while for $\sigma > \sigma_{rev}$ we find the point of optimal operation along the curve of intersection of the $\sigma = \sigma_{max}$ surface with the plane (6.19) for maximum efficiency and effectiveness or with the plane (6.26) for minimum entropy production and loss of availability.

C. Case F: Maximum profit

As Eq. (6.35) shows, the profit depends on our parameters exactly as the power provided that T_1^{ex} and T_2^{ex} are replaced by T_1^{eff} and T_2^{eff}. Thus, profit achieves its abolute maximum of

$$\Pi = P_W \cdot \tfrac{1}{4}\kappa(\sqrt{T_1^{eff}} - \sqrt{T_2^{eff}})^2 \tag{7.9}$$

anywhere on the line

$$\tau_1 = \tau_2 = \frac{\sigma}{\kappa}\ \frac{\sqrt{T_1^{eff}} + \sqrt{T_2^{eff}}}{\sqrt{T_1^{eff}} - \sqrt{T_2^{eff}}}\ . \tag{7.10}$$

The constrained optimizations also proceed analogously. Let $\hat{\sigma}$ and $\hat{\tau}$ denote the values of σ and τ at the point of intersection of the line (7.10) with the $\sigma = \sigma_{max}$ surface (see Fig. 10). For optimization with constrained $\sigma < \hat{\sigma}$, we choose τ so that (7.9) holds. For $\sigma > \hat{\sigma}$, we choose the largest τ, which is as close as we can get to satisfying Eq. (7.9). This gives points along the intersection of the plane (6.36) with the surface $\sigma = \sigma_{max}$. Similarly, for constrained τ we operate optimally on the line (7.10) if $\tau < \hat{\tau}$, and on the intersection of Eq. (6.36) and $\sigma = \sigma_{max}$ for $\tau > \hat{\tau}$. Note that for all our optimizations the conclusions of Sec. VI remain valid: As $P_A/P_W \to 0$, maximum profit operation approaches maximum power operation, while as $P_A/P_W \to 1$, maximum profit operation approaches minimum entropy production operation.

VIII. THE PRODUCTION FUNCTION

The information in Eq. (7.10) can be used to give the production function for our process. Though analytically possible in general, finding the production function is very cumbersome unless the temperature of the environment coincides with the temperature of one of our reservoirs, say $T_{en} = T_2^{ex}$. Then $\eta_2 = 0$ and $\eta_1 = \eta_{rev}$. Then inverting Eq. (3.16) using also [cf. (7.10)]

$$\tau_1 = \tau/2 \tag{8.1}$$

leads to

$$\frac{\sigma}{\tau} = \frac{1}{[T_1^{ex}/(Q_1/\tau)] - \frac{2}{\kappa}}\ , \tag{8.2}$$

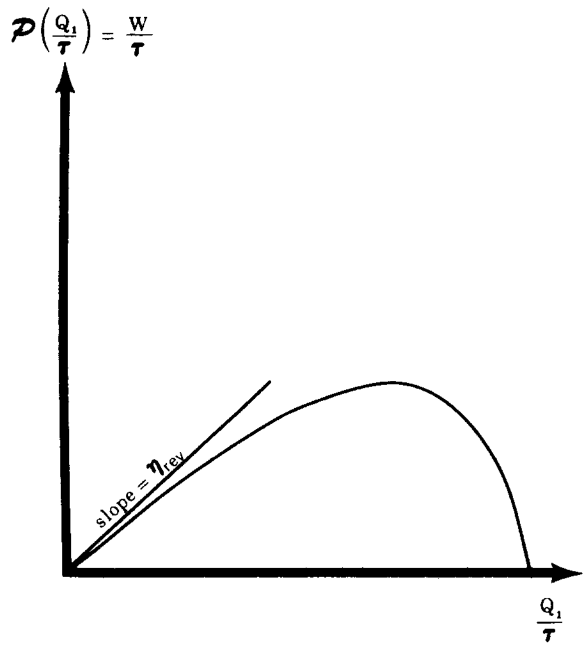

FIG. 11. The production function [Eq. (8.4)] for a Newton's law Carnot engine. The output (power) is plotted as a function of the input (rate of heating).

which when substituted into [cf. Eqs. (2.3), (3.16), and (3.17)]

$$\frac{W}{\tau} = \frac{T_1^{ex}\left(\frac{\sigma}{\tau}\right)}{1 + \frac{2}{\kappa}\left(\frac{\sigma}{\kappa}\right)} - \frac{T_2^{ex}\left(\frac{\sigma}{\tau}\right)}{1 - \frac{2}{\kappa}\left(\frac{\sigma}{\tau}\right)} \tag{8.3}$$

gives the production function $\mathcal{P}$ as

$$\frac{W}{\tau} = \mathcal{P}\left(\frac{Q_1}{\tau}\right) = \left(\frac{Q_1}{\tau}\right)\left[\frac{T_1^{ex} - T_2^{ex} - \frac{4}{\kappa}\left(\frac{Q_1}{\tau}\right)}{T_1^{ex} - \frac{4}{\kappa}\left(\frac{Q_1}{\tau}\right)}\right] \tag{8.4}$$

pictured in Fig. 11. For maximum profit the slope of this curve must be

$$\frac{\partial W/\tau}{\partial Q_1/\tau} = \frac{P_{Q1}}{P_W} = \frac{P_A\,\eta_{rev}}{P_W}\ , \tag{8.5}$$

where we have introduced the symbol $P_{Q_1} \equiv P_A\,\eta_{rev}$, which may be interpreted as the price of heat Q_1. [The relation (8.5) is obtained by maximizing the profit $(P_W W/\tau) - (P_{Q_1} Q_1/\tau)$ with respect to the input Q_1 (see, for example, the books cited in Ref. 14).] Thus, given prices locate us on the curve in Fig. 10 by specifying the slope at the point of maximal profit operation. Note again that as $P_A/P_W \to 0$ maximum profit operation approaches maximum power operation, while as $P_A/P_W \to 1$ maximum profit operation approaches minimum entropy production and maximum efficiency operation at the points where the slope vanishes and where it equals the reversible efficiency.

Since in our model the engine uses only a single input, a given cost or a given level of output contrains us to a single point on the curve (8.4). This makes the other economic optimizations discussed in Sec. III uninteresting.

IX. CONCLUSIONS

In the sections above, we found the optimal operations of a Carnot cycle for various choices A–F of optimal operation and subject to various constraints. The results are summarized in Figs. 7–10.

Besides possible implications for engine design,[2] analyses such as the one above can be used to predict in-principle thermodynamic bounds to the quality of operation of a thermodynamic process within a given technology. As shown above, such bounds can be found over very large classes of processes independently of specific designs and working fluids. Though no such analyses have been carried out for important industrial processes, such analyses could be important aides in allocating research and development budgets with the objective of decreasing energy consumption. Our analysis for Carnot cycles in Newton's law thermodynamics serves as a paradigm for such calculations and reveals serveral useful tricks for simplifying such analyses.

The first useful trick is the reduction from optimization in an infinite dimensional space to optimization in a finite dimensional space, i.e., from choosing the optimal value of $T(t)$ for each time $0 \le t \le \tau$, to choosing only the values of three numbers: τ_1, τ_2, and σ. Any reasonable objective function for a thermodynamic process will depend only on net effects of the process. The space of such net effects is finite dimensional since it can be parametrized by the initial and final states, the time elapsed during the process, and parameters, such as κ, which represent the part of the technology we take to be given.[24] A reduction to an optimization in this space of net effects is probably possible generally[25] and allows comparison of optimal operations for different objective functions. Following the reduction, the problem may be further simplified by using the methods of monotonic substitution and of successive optimization (of which one branch variations are an important special case).

Newton's law thermodynamics defines, perhaps, the simplest technology in which such optimizations can be performed and provides a wealth of analytically soluble problems.[1,2] The freedom of making adiabatic jumps eliminates all constraints except constraints on the change in entropy and on the elapsed time. When combined with the technique of local optimizations, i.e., optimizations of only one branch at a time, this freedom can considerably simplify the calculations required for the optimization.[26]

The importance of the present results in Newton's law thermodynamics would be greatly enhanced if it could be shown that optimal processes with only a Newton's law type irreversibility always perform better than optimal processes which have this same irreversibility (with the same κ's) plus some other mode of irreversible entropy production, e.g., friction. This conjecture[27] has great intuitive appeal and one is tempted to accept it on physical grounds, though a formal argument is desirable.

Our results concerning the reduction to Carnot space (τ_1, τ_2, σ) are valid even if we replace the linear Newton's law (1.1) by an arbitrary rate law

$$dQ/dt = f(T^{ex}, T) , \tag{9.1}$$

which vanishes for $T = T^{ex}$ and which is monotonic in T. Although the algebraic expressions for the optimal cycles will be different and our boundaries and optima will no longer be lines and hyperbolas, the qualitative features of $\overline{C}$ and of the sets of optimal operations is the same for a large class of rate laws.[28]

As mentioned in Ref. 23, if we alter the objective function ΔS_u (case D) and the objective function ΔA_u (case E) to the corresponding rates $\Delta S_u/\tau$ (case D′) and $\Delta A_u/\tau$ (case E′), our definition of Carnot space implies that $\overline{C}'$ is two dimensional since all the objective functions depend only on the ratios σ/τ_1 and σ/τ_2. Thus, $\overline{C}'$ can be identified with the projective two space formed from (τ_1, τ_2, σ). This projective symmetry comes from the fact that an engine carrying entropy $\sigma/2$ in time $\tau/2$ performing two cycles is equivalent to one carrying entropy σ in time τ performing one cycle. We chose the objective functions D and E rather than D′ and E′ to eliminate this symmetry. This is desirable for several reasons. First, the boundary $\sigma = \sigma_{max}$ does not share this projective symmetry. Further, such symmetry would also be destroyed by the presence of depreciation cost terms in the economic objective functions. For example, let us take depreciation cost to be proportional to the number of cycles

$$C_\tau = P_\tau / \tau . \tag{9.2}$$

While the profit without counting depreciation is radially constant, i.e., constant along any line through the origin, the depreciation cost is monotonic decreasing as we move out along such lines. It therefore follows that we should move out as far as possible, i.e., optimal profit operation including the depreciation cost (9.2) will lie on the boundary $\sigma = \sigma_{max}$. For small P_τ, such optima will lie near the point of intersection of the $\sigma = \sigma_{max}$ surface with the line (7.11).

In Secs. VI–VIII we saw repeatedly that economic and thermodynamic optimizations of our energy conversion process converged in the limits $P_A/P_W \to 0$, 1. Specifically, we saw that when the profit margin for energy conversion is small, the maximum profit operation is near the minimum loss of availability operation, while when availability is very cheap compared to the price of work, the maximum profit operation is near the maximum power operation. For intermediate P_A/P_W, we find the implication that *any publically desirable efficiency can be made profit optimal by regulating the ratio P_A/P_W of input and output prices*. A related conclusion for energy consuming production processes was found by R. S. Berry, P. Salamon, and G. Heal.[29]

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Procaccia, D. Soda, J. W. Stout, and S. W. Sullivan for stimulating discussions related to this work. We also wish to thank Morton Rubin for pointing out some slight inaccuracies in an earlier version of this paper.

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[13]For a general proof of the equivalence of these two criteria, see the Appendix in Ref. 1.
[14]Short-run and long-run economic optimizations refer to optimizations on different time scales. In the short run some factors of production are fixed and not subject to optimization. In the long run, however, the producer is completely free to vary any part of the production process, including building a new factory. For a fuller discussion of these terms and other economic matters see, for example, J. M. Henderson and R. E. Quandt, *Microeconomic Theory* (McGraw-Hill, London, 1958); E. P. DeGarmo and J. R. Canada, *Engineering Economy* (Macmillan, New York, 1973).
[15]Each of these three optimizations can be done on any time scale, i.e., for short-run and long-run optimizations. The time scales merely specify which inputs are constrained to be constant for the specific optimization. Further, there is a fourth optimization (usually implicit) in the production function itself, as it is defined to be the most product which can be produced using the given quantities of inputs. For further details see Refs. 14, 16, and 17.
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[20]Our definition of equivalent operations defines an equivalence relation on the set of control programs $T(t)$. $\overline{C}$ is a subset of the quotient set defined by this relation.
[21]For a refrigerator σ would be negative. Our analysis considers only engines so that σ is positive.
[22]Interior optima are optima at interior points of the space over which the optimization is carried out (as opposed to optima which lie on the boundaries). For differentiable objective functions, interior optima can be found from the condition that the derivative of the objective function vanishes.
[23]It might be preferable to change objective functions ΔS_u (Case D) and ΔA_u (Case E) to $\Delta S_u/\tau$ (Case D′) and $\Delta A_u/\tau$ (Case E′), in which case operations (τ_1, τ_2, σ) and $(\lambda\tau_1, \lambda\tau_2, \lambda\sigma)$ become equivalent for all $\lambda>0$. Thus, all the objective functions become constant on lines through the origin and $\overline{C}$ becomes a projective 2 space. The only equation which lacks projective symmetry is the equation of the boundary $\sigma=\sigma_{max}(\tau_1, \tau_2)$. We have chosen objective functions D and E rather than objective functions D′ and E′ to eliminate this symmetry and show more clearly the role of the boundary. For some applications it may prove desirable to use the objective functions D′ and E′.
[24]Newton's law thermodynamics assumes a particularly simple technology represented by the single parameter κ. More complicated models would require more parameters and would better approximate real technologies.
[25]The feasibility of this conjecture was pointed out to us by R. S. Berry.
[26]Compare, for example, Refs. 4 and 5, where some of our results are obtained by other means.
[27]The unconscious use of this conjecture in our arguments was pointed out to us by O. Kafri.
[28]The material in this paragraph has benefited from discussion with C. A. Hutchison Jr. Similar considerations were followed in Ref. 5.
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