New Journal of Physics
The open-access journal for physics

Nonclassical time correlation functions in continuous quantum measurement

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New Journal of Physics **14** (2012) 013009 (20pp)
Received 5 October 2011
Published 6 January 2012
Online at http://www.njp.org/
doi:10.1088/1367-2630/14/1/013009

Abstract. A continuous projective measurement of a quantum system often leads to a suppression of the dynamics, known as the Zeno effect. Alternatively, generalized nonprojective, so-called 'weak' measurements can be carried out. Such a measurement is parameterized by its strength parameter that can interpolate continuously between the ideal strong measurement with no dynamics—the strict Zeno effect, and a weak measurement characterized by almost free dynamics but blurry observations. Here we analyze the stochastic properties of this uncertainty component in the resulting observation trajectory. The observation uncertainty results from intrinsic quantum uncertainty, the effect of measurement on the system (backaction) and detector noise. It is convenient to separate the latter, system-independent contribution from the system-dependent uncertainty, and this paper shows how to accomplish this separation. The system-dependent uncertainty is found in terms of a quasi-probability, which, despite its weaker properties, is shown to satisfy a weak positivity condition. We discuss the basic properties of this quasi-probability with special emphasis on its time correlation functions as well as their relationship to the full correlation functions along the observation trajectory, and illustrate our general results with simple examples. We demonstrate a violation of classical macrorealism using the fourth-order time correlation functions with respect to the quasi-probability in the two-level system.

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Contents

1. Introduction

The continuous projective von Neumann quantum measurement [1] leads to a suppression of the dynamics, known as the quantum Zeno effect (QZE) [2]. To escape this problem, modern quantum measurement theory offers generalizations of the projective measurement to the so-called positive operator-valued measures (POVM) [3, 4], where a projection is replaced by a softer operation such as a Kraus operator [5]. Such operators can describe not only projective measurements but also weak measurement, in which case the action of the POVM leaves the state almost unchanged. By virtue of the Naimark theorem [6], POVMs are equivalent to projective measurements in an extended Hilbert space that includes additional detector degrees of freedom. The effect of a continuous application of Kraus operators, which correspond to a time-continuous measurement, can be described by stochastic evolution equations such as Lindblad-type equations [7] for the system density matrix or Langevin equations for individual system trajectories, physically describing irreversible effects such as decoherence and decay affected by the measurement process.

Weak measurements [8] make it possible to escape the QZE by paying a price in terms of an imperfect detection. In the extreme case the dynamics of the system is (almost) free but the measurement outcome is obscured by a large detection noise. This is similar to the problem of a quantum linear amplifier, which can amplify both complementary noncommuting observables, like $\hat{x}$ and $\hat{p}$, but only if accompanied by a large noise [9]. The interpretation of weak measurements of correlation functions is sometimes paradoxical: one must either accept unusually large values of the physical quantity [8] or replace the probability by a quasi-probability [10]. Weak measurements are also very useful in quantum feedback protocols [11].

The QZE lies at the strong limit of a spectrum of measurements whose strong/weak character can be changed continuously [12–19], using e.g. a Gaussian POVM [20, 21]. The Gaussian POVM is also the key element of the continuous collapse interpretation of quantum mechanics [22]. These models lead to various types of expressions for time correlation functions [23]. Here, for the first time we use Gaussian POVM for continuous measurement to describe higher-order symmetrized time correlation functions. Such functions are known for the two-time case of a harmonic system [20] or in the weak measurement limit for the two-level system [24] (but not in general) and are necessary to explain the recent experiment that shows nonclassical behavior of time correlation functions in a two-level system [25]. The calculations are facilitated by making a deconvolution of the outcome time trace probability into the probability component associated with the white detection noise and a quasi-probability that describes the intrinsic system uncertainty. Such a deconvolution has the advantage that we can make use of basic properties of the quasi-probability, e.g. the weak positivity [26], which states that the second-order correlation function matrix is positive definite. Our scheme provides a unified and concise treatment of weak measurements and the QZE, pointing out the general trade-off between measurement and decoherence. By comparing the average signal to the associated noise we also establish limits on the uncertainty of the outcome and its dependence on measurement characteristics.

The time correlation functions obtained by our approach provide a convenient route for the analysis of uncertainty properties of systems undergoing weak measurements. Taking a two-level system as an example, a single nondemolishing measurement of an observable not commuting with the Hamiltonian is not possible in both time and frequency domains although the latter gives a better signal-to-noise ratio. Although this is intuitively clear, using our approach, it is possible to establish and compare bounds on the outcome uncertainty. For another simple example, the continuous position measurement of a harmonic oscillator, we show that the same measurement procedure does not lead to the QZE. Instead, the continuous measurement leads to unbounded growth in noise, in agreement with the general observation [21] and in analogy to the anti-Zeno effect [27].

The proposed separation has another important consequence. If we assume classical macrorealism in quantum mechanics, then the statistics of the outcomes with the detection noise subtracted in the limit of noninvasive measurement should correspond to a positive definite probability. In contrast, we show that the macrorealism assumption is violated by demonstrating that our quasi-probability is somewhere negative. Such violation has recently been demonstrated experimentally [25]. In fact, if we additionally assume dichotomy or boundedness of the quantum outcomes, the violation can occur already on the level of second-order correlations of a single observable as shown by Leggett and Garg and [28] others [24], but also indirectly, subtracting the unavoidable (and necessarily divergent) noise [29]. However, as follows from weak positivity, without these additional assumptions, second-order correlations are not sufficient to violate macrorealism. Instead, one needs at least fourth-order averages to see this violation. In this paper, we demonstrate that a special fourth-order correlation function in the two-level system, reminiscent of the Leggett–Garg proposal [28], can reveal the negativity of the quasi-probability in this case and consequently can be used to violate macrorealism, without having to make any additional assumptions.

This paper is organized as follows. We first define the continuous Gaussian POVM and obtain the probability distribution for the continuously measured observable. We then make the deconvolution of this probability into detection noise and a quasi-probability and introduce a formalism for evaluating time correlation functions. With that we are able to prove the weak positivity. We show that the time evolution associated with the quasi-probability can be formulated either as a quantum Langevin equation driven by a white Gaussian noise or as a Lindblad-type master equation for the nonselective system density matrix. We also show how the required time correlation functions can be calculated from these stochastic equations. Next, we demonstrate the general trade-off between dynamics and measurement, taking a two-level system as an example, and discuss the behavior of the average signal and the noise in these prototype systems. Then we construct the Leggett–Garg inequality without assuming dichotomy or boundedness of the measurement variable. Finally, for completeness, we discuss the harmonic oscillator case and show how and when the Zeno effect emerges within our formalism. Several instructive proofs of formulae are presented in the appendices.

2. Quasi-probability and weak positivity

We begin by introducing a general scheme of continuous measurement and describe its properties. For a given system characterized by a Hamiltonian $\hat{H}$ and an initial system state given by a density matrix $\hat{\rho}$, we consider the measurement of one, generally time-dependent, observable $\hat{A}$. A description amenable to continuous interpolation between hard and soft measurements can be formulated in terms of the Kraus operators [5, 8]. We assume a Gaussian form of the Kraus operators, whereupon the state of the system following a single instantaneous measurement is given by

$$\hat{\rho}_1(a) = \hat{K}(a)\hat{\rho}\hat{K}(a), \tag{1}$$

$$\hat{K}(a) = (2\bar{\lambda}/\pi)^{1/4}\mathrm{e}^{-\bar{\lambda}(a-\hat{A})^2}. \tag{2}$$

Note that in (1) the non-negative definite operators $\hat{\rho}$ and $\hat{\rho}_1$ represent the states of the system just before and just after the measurement. The probability that the measurement of $\hat{A}$ gives the outcome a is given by [3]

$$P(a) = \mathrm{Tr}\,\hat{\rho}_1(a), \tag{3}$$

which is normalized, $\int \mathrm{d}a\, P(a) = 1$. The Kraus operator (2) depends on the parameter $\bar{\lambda}$, which characterizes the weakness of the measurement. For $\bar{\lambda} \to \infty$, we recover a strong, projective measurement with an exact result but a complete destruction of coherence, while $\bar{\lambda} \to 0$ corresponds to a weak measurement with almost no influence on the state of the system, $\hat{\rho}_1(a) \sim \hat{\rho}$, but a very large measurement uncertainty of the order of $\sim 1/\bar{\lambda}$. The probability distribution (3) is consistent with the projective measurement scheme, namely $\langle a \rangle = \mathrm{Tr}\,\hat{A}\hat{\rho}$.

Let us imagine that a continuous sequence of meters interacts with the system. The meters are prepared with a Gaussian wave function, the interaction is proportional to the product of the system observable $\hat{A}$ and the meter momentum, and the position of each meter is read out after the interaction. The post-interaction position of the meters is the measurement result $a(t)$ [4, 19, 20].

Repeated measurements of this type can be described by applying such Kraus operators sequentially, separated by time steps Δt. In what follows we make the reasonable assumption that for a given measuring device ('meter') the weakness parameter $\bar{\lambda}$ is inversely proportional to the measurements frequency, i.e.

$$\bar{\lambda} = \lambda \Delta t \tag{4}$$

with constant λ. In the continuum limit, $\bar{\lambda}$, $\Delta t \to 0$, we obtain (appendix A) the Kraus operator as a functional of $a(t)$

$$\hat{K}_{\mathrm{h}}[a(t)] \equiv \mathrm{e}^{(\mathrm{i}/\hbar)\hat{H}t}\,\hat{K}\,[a\,(t)] = C\mathcal{T}\,\mathrm{e}^{-\int \lambda(a(t)-\hat{A}(t))^2 \mathrm{d}t}, \tag{5}$$

where $a(t)$ is the measurement outcome, $\hat{A}(t)$ is the operator $\hat{A}$ in the Heisenberg representation with respect to the Hamiltonian $\hat{H}$, $\hat{A}(t) = \exp(\mathrm{i}\hat{H}t/\hbar)\hat{A}\exp(-\mathrm{i}\hat{H}t/\hbar)$, $\mathcal{T}$ denotes time ordering (later times on the left) and C is a normalization factor. Note that $\hat{K}_{\mathrm{h}}[a]$ is the Heisenberg representation of $\hat{K}[a]$. The analogue of (3) is the functional probability

$$P[a] = \mathrm{Tr}\,\hat{K}^{\dagger}[a]\hat{K}[a]\hat{\rho}, \tag{6}$$

which satisfies the normalization $\int Da\,P[a] = 1$. Whenever some functional measure D is introduced here, we tacitly include all proper normalization factors in it.

It is convenient to write (5) as a Fourier transform

$$\hat{K}_{\mathrm{h}}[a] = \int D\phi\,\mathcal{T}\,\mathrm{e}^{\int \mathrm{d}t[\mathrm{i}\phi(t)(\hat{A}(t)-a(t))-\phi^2(t)/4\lambda]} \tag{7}$$

so that

$$P[a] = \mathrm{Tr}(\hat{K}\,[a]\,\hat{\rho}\hat{K}\,[a]) = \mathrm{Tr}\left(\int D\phi_+\mathcal{T}\,\mathrm{e}^{\int \mathrm{d}t[\mathrm{i}\phi_+(t)(\hat{A}(t)-a(t))-\phi_+^2(t)/4\lambda]}\right.$$
$$\left.\times\,\hat{\rho}\int D\phi_-\tilde{\mathcal{T}}\,\mathrm{e}^{\int \mathrm{d}t[\mathrm{i}\phi_-(t)(\hat{A}(t)-a(t))-\phi_-^2(t)/4\lambda]}\right), \tag{8}$$

where $\tilde{\mathcal{T}}$ denotes inverse time ordering (later times on the right). Changing integration variables according to $\chi = \phi_+ + \phi_-$ and $\phi = (\phi_+ - \phi_-)/2$, we can write

$$P[a] = \int D\phi\,\mathrm{e}^{-\int \mathrm{d}t\phi^2(t)/2\lambda}\int D\chi\,\mathrm{e}^{-\int \mathrm{d}t\chi^2(t)/8\lambda}\,\mathrm{e}^{-\int \mathrm{i}\chi(t)a(t)\mathrm{d}t}$$
$$\times\mathrm{Tr}\,\mathcal{T}\,\mathrm{e}^{\int \mathrm{i}(\chi(t)/2+\phi(t))\hat{A}(t)\mathrm{d}t}\,\hat{\rho}\,\tilde{\mathcal{T}}\,\mathrm{e}^{\int \mathrm{i}(\chi(t)/2-\phi(t))\hat{A}(t)\mathrm{d}t}. \tag{9}$$

The last line can be written alternatively as (see appendix B)

$$\mathrm{Tr}\,\mathcal{T}\,\mathrm{e}^{\mathrm{i}\int \chi(t)\hat{A}_{\phi}(t)\mathrm{d}t/2}\,\hat{\rho}\,\tilde{\mathcal{T}}\,\mathrm{e}^{\mathrm{i}\int \chi(t)\hat{A}_{\phi}(t)\mathrm{d}t/2}, \tag{10}$$

where $\hat{A}_{\phi}(t)$ denotes the operator $\hat{A}$ in a modified Heisenberg picture, namely with respect to the Hamiltonian $\hat{H} - \hbar\phi(t)\hat{A}$.

Equations (9) and (10) describe the outcome of the continuous measuring process in terms of the probability distribution functional $P\,[a\,(t)]$ of the observation function $a(t)$. This distribution reflects the quantum uncertainty, the modified system time evolution caused by the measurement (the backaction effect) and the uncertainty associated with the weak measurement that can be thought of as reflecting detector noise. A more transparent view of these contributions is obtained by separating the latter, system-independent contribution from the system-dependent effects. This is achieved by considering the moment generating functional $M[\chi] = \mathrm{e}^{S[\chi]}$, where $S\,[\chi]$ is the cumulant generating functional (CGF), given by

$$M[\chi] = \mathrm{e}^{S[\chi]} = \int Da\,\mathrm{e}^{\mathrm{i}\int \chi(t)a(t)}P[a]$$
$$= \mathrm{e}^{-\int \mathrm{d}t\chi^2(t)/8\lambda}\int D\phi\,\mathrm{e}^{-\int \mathrm{d}t\phi^2(t)/2\lambda}\mathrm{Tr}\,\mathcal{T}\mathrm{e}^{\mathrm{i}\int \chi(t)\hat{A}_{\phi}(t)\mathrm{d}t/2}\,\hat{\rho}\,\tilde{\mathcal{T}}\mathrm{e}^{\mathrm{i}\int \chi(t)\hat{A}_{\phi}(t)\mathrm{d}t/2}. \tag{11}$$

The CGF can be divided into two parts $S[\chi] = S_{\rm d}[\chi] + S_{\rm q}[\chi]$ with

$$S_{\rm d}[\chi] = -\int {\rm d}t\,\chi^2(t)/8\lambda \tag{12}$$

and

$$e^{S_{\rm q}[\chi]} = \int D\phi\; e^{-\int {\rm d}t\phi^2(t)/2\lambda} {\rm Tr}\,\mathcal{T} e^{{\rm i}\int \chi(t)\hat{A}_\phi(t){\rm d}t/2}\,\hat{\rho}\,\tilde{\mathcal{T}} e^{{\rm i}\int \chi(t)\hat{A}_\phi(t){\rm d}t/2}. \tag{13}$$

Note that $S[0] = S_{\rm d}[0] = S_{\rm q}[0] = 0$. On the level of probabilities this decomposition corresponds to the convolution

$$P[a] = \int Da'\,P_{\rm d}[a-a']P_{\rm q}[a'], \tag{14}$$

where

$$P_{\rm d}[a] = \int D\chi\; e^{\int {\rm d}t(\chi(t)a(t)/{\rm i} - \chi^2(t)/8\lambda)} \propto e^{-2\lambda\int a^2(t){\rm d}t} \tag{15}$$

corresponds to a Gaussian noise with zero average and correlation $\langle a(t)a(t')\rangle_{\rm d} = \delta(t-t')/4\lambda$ that may be interpreted as the noise associated with the detector, and where

$$P_{\rm q}[a] = \int D\chi\; e^{-\int {\rm i}\chi(t)a(t){\rm d}t} e^{S_{\rm q}[\chi]} \tag{16}$$

is a distribution associated with the intrinsic system uncertainty as well as the measurement backaction. It is normalized, $\int Da\; P_{\rm q}[a] = 1$, but not necessarily positive, and will be referred to as a quasi-probability [10, 30, 31]. In the limit of weak, noninvasive measurement, $\lambda \to 0$, $P_{\rm d}$ diverges while $P_{\rm q}$ has a well-defined limit

$$e^{S_{\rm q}[\chi]} \xrightarrow{\lambda\to 0} {\rm Tr}\,\mathcal{T} e^{{\rm i}\int \chi(t)\hat{A}(t){\rm d}t/2}\,\hat{\rho}\,\tilde{\mathcal{T}}\, e^{{\rm i}\int \chi(t)\hat{A}(t){\rm d}t/2}. \tag{17}$$

Consider now this distribution $P_{\rm q}$. First note that while it is not a real probability functional, it is possible to calculate moments $\langle\rangle$ and cumulants $\langle\langle\rangle\rangle$ with respect to this measure as partial derivatives of the quasi-CGF, respectively,

$$\begin{aligned} \langle a(t_1)\cdots a(t_n)\rangle_{\rm q} &= \left.\frac{\delta^n \exp S_{\rm q}[\chi]}{\delta{\rm i}\chi(t_1)\cdots\delta{\rm i}\chi(t_n)}\right|_{\chi=0}, \\ \langle\langle a(t_1)\cdots a(t_n)\rangle\rangle_{\rm q} &= \left.\frac{\delta^n S_{\rm q}[\chi]}{\delta{\rm i}\chi(t_1)\cdots\delta{\rm i}\chi(t_n)}\right|_{\chi=0}. \end{aligned} \tag{18}$$

In particular, for $t_n \geqslant \cdots \geqslant t_2 \geqslant t_1$,

$$\langle a(t)\rangle_{\rm q} = \int D_\lambda\phi\;{\rm Tr}[\hat{A}_\phi(t)\hat{\rho}], \tag{19a}$$

$$\langle a(t_1)a(t_2)\rangle_{\rm q} = \int D_\lambda\phi\;{\rm Tr}[\{\hat{A}_\phi(t_1), \hat{A}_\phi(t_2)\}\hat{\rho}]/2, \tag{19b}$$

$$\begin{aligned} \langle a(t_1)a(t_2)a(t_3)\rangle_{\rm q} &= \int D_\lambda\phi\;{\rm Tr}[\{\hat{A}_\phi(t_1), \{\hat{A}_\phi(t_2), \hat{A}_\phi(t_3)\}\}\hat{\rho}]/4, \\ \langle a(t_1)\cdots a(t_n)\rangle_{\rm q} &= \int D_\lambda\phi\;{\rm Tr}[\{\hat{A}_\phi(t_1), \{\hat{A}_\phi(t_2), \cdots \hat{A}_\phi(t_n)\}\cdots\}\hat{\rho}]/2^{n-1} \end{aligned} \tag{19c}$$

(see appendix C), where we have defined $D_\lambda\phi = D\phi\, \mathrm{e}^{-\int \mathrm{d}t\, \phi^2(t)/2\lambda}$. Here and below we use the standard notation $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ and $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$.

Secondly, from equation (19b) follows the important so-called *weak positivity* property of second-order correlations [26]

$$\langle F^2[a]\rangle_{\mathrm{q}} = \int D\phi\, \mathrm{e}^{-\int \mathrm{d}t \phi^2(t)/2\lambda}\, \mathrm{Tr}\, F^2[\hat{A}_\phi]\hat{\rho} \geqslant 0 \tag{20}$$

for $F[a] = \int \mathrm{d}t\,(f(t)a(t) + g(t))$ and arbitrary functions f and g. It can be interpreted as a generalization of the Robertson–Schrödinger uncertainty principle [32]. This property has an important implication that no test based solely on maximally second-order correlations can reveal the negativity of the quasi-probability. First- and second-order correlations can be represented by a completely classical, positive Gaussian probability distribution

$$P'_{\mathrm{q}}[a] \propto \exp\left(-\int \mathrm{d}t\, \mathrm{d}t'\delta a(t) f^{-1}(t,t')\delta a(t')/2\right), \tag{21}$$

where $\delta a(t) = a(t) - \langle a(t)\rangle_{\mathrm{q}}$, $f(t,t') = \langle \delta a(t)\delta a(t')\rangle_{\mathrm{q}}$ and f^{-1} is its inverse defined by $\int \mathrm{d}t f(t',t) f^{-1}(t,t'') = \delta(t'-t'')$. The weak positivity guarantees that both f and f^{-1} are positive definite and, consequently, P'_{q} is a correct real probability distribution. To check that P_{q} differs from P'_{q} and demonstrate its negativity, one needs higher-order correlations or additional assumptions (e.g. boundedness or dichotomy of a as it happens in the Leggett–Garg inequality [28]).

To end this section, we consider the special case when the Hamiltonian commutes with $\hat{A}$ (or the noncommuting part is negligible during the interesting timescale). Furthermore, let us take the initial state of the system to be an eigenstate $|a\rangle$ of $\hat{A}$, i.e. $\hat{\rho}\,(t=0) = |a\rangle\langle a|$, $\hat{A}|a\rangle = a|a\rangle$. Consider a measurement carried out during the time interval t_0,

$$\bar{a} = (1/t_0)\int_0^{t_0} \mathrm{d}t\; a(t). \tag{22}$$

In this case, we find (appendix D) that $\langle \bar{a}\rangle = \langle \bar{a}\rangle_{\mathrm{q}} = a$ and $\langle(\delta\bar{a})^2\rangle = \langle(\delta\bar{a})^2\rangle_{\mathrm{d}} = 1/4\lambda t_0$ with $\delta X = X - \langle X\rangle$. We can see the intuitively expected effect of an increasing measurement duration to lead to an improved signal-to-noise ratio with time, which goes as

$$\frac{\langle \bar{a}\rangle}{\sqrt{\langle(\delta\bar{a})^2\rangle}} = 2a\sqrt{\lambda t_0}\,. \tag{23}$$

Thus, even the weakest measurement (small λ) turns into a strong one if performed often enough for a sufficiently long time.

3. Representation by stochastic evolution equations

Turning back to the general case, we note first that the correlation functions associated with the quasi-probability $P_{\mathrm{q}}[a(t)]$, given by equations (19*a*)–(19*c*), can be calculated from the Heisenberg equations

$$\mathrm{d}\hat{B}_\phi(t)/\mathrm{d}t = (\mathrm{i}/\hbar)[\hat{H}_\phi(t) - \phi(t)\hat{A}_\phi(t), \hat{B}_\phi(t)]\,, \tag{24}$$

where $\hat{A}$ represents the measured variable while $\hat{B}$ is any system operator. In particular,

$$\begin{aligned} \mathrm{d}\hat{A}_\phi(t)/\mathrm{d}t &= (\mathrm{i}/\hbar)[\hat{H}_\phi(t), \hat{A}_\phi(t)], \\ \mathrm{d}\hat{H}_\phi(t)/\mathrm{d}t &= \phi(t)\, \mathrm{d}\hat{A}_\phi(t)/\mathrm{d}t. \end{aligned} \tag{25}$$

We can solve these equations for a general stochastic trajectory $\phi(t)$ and then take the averages as defined by equations (19*a*)–(19*c*), over a Gaussian distribution of such trajectories. The correlation functions obtained in this way coincide with the ones derived directly from the CGF equation (16). If $\hat{H} = \hat{p}^2/2m + V(\hat{x})$, with $[\hat{x}, \hat{p}] = \mathrm{i}\hbar\hat{1}$, and $\hat{A} = \hat{x}$ is the position operator, the Heisenberg equations for $\hat{x}_\phi$ and $\hat{p}_\phi$ are

$$\begin{aligned} \hat{H}_\phi &= \hat{p}_\phi^2/2m + V(\hat{x}_\phi), \\ \frac{\mathrm{d}\hat{x}_\phi}{\mathrm{d}t} &= \hat{p}_\phi/m, \\ \frac{\mathrm{d}\hat{p}_\phi}{\mathrm{d}t} &= (\mathrm{i}/\hbar)[V(\hat{x}_\phi), \hat{p}_\phi] + \hbar\phi(t) = -\frac{\partial V(\hat{x}_\phi)}{\partial \hat{x}_\phi} + \hbar\phi(t). \end{aligned} \tag{26}$$

Equation (26) is a quantum Langevin equation in which the quantum dynamics is augmented by a zero centered white Gaussian noise, $\langle\phi(t)\rangle = 0$, $\langle\phi(t)\phi(t')\rangle = \lambda\delta(t-t')$. Closed-form solutions of this equation can be obtained for the harmonic oscillator, a case we discuss below.

Alternatively, the stochastic dynamics affected by the continuous measurement process may be described by a Lindblad-type master equation [7] for the nonselective system density matrix. The latter is defined by

$$\hat{\tilde{\rho}}(t) = \int_{a(0)}^{a(t)} Da\,\hat{\rho}[a] = \int_{a(0)}^{a(t)} Da\,\hat{K}[a]\hat{\rho}\hat{K}^\dagger[a], \tag{27}$$

where the integral is over all observation trajectories between times 0 and t. It is shown (appendix E) to evolve according to (using a Liouville superoperator $\check{L}$)

$$\frac{\mathrm{d}\hat{\tilde{\rho}}}{\mathrm{d}t} = \check{L}\hat{\tilde{\rho}} := [\hat{H}, \hat{\tilde{\rho}}]/\mathrm{i}\hbar - \lambda[\hat{A}, [\hat{A}, \hat{\tilde{\rho}}]]/2. \tag{28}$$

In the representation of eigenstates of $\hat{A}$,

$$\hat{\tilde{\rho}} = \sum_{a,a'} \tilde{\rho}_{aa'}|a\rangle\langle a'|, \tag{29}$$

$$\check{L}\tilde{\rho}_{a,a'} = \frac{1}{\mathrm{i}\hbar}\sum_b (H_{ab}\tilde{\rho}_{ba'} - \tilde{\rho}_{ab}H_{ba'}) - \lambda(a-a')^2\tilde{\rho}_{aa'}, \tag{30}$$

showing, as is well known [4] and as may be intuitively expected, that the measurement damps the off-diagonal terms ($a \neq a'$) with the rate proportional to the measurement strength. Note that if some eigenvalues a are degenerate, then the corresponding off-diagonal elements of $\hat{\tilde{\rho}}$ are not damped.

Together with the Liouville–Lindblad superoperator $\check{L}$, we define the corresponding evolution superoperator $\check{U}(a,b) = \mathcal{T}\exp\int_b^a \check{L}\mathrm{d}t$. It can be then shown (appendix F) that the correlation functions (19*a*)–(19*c*) are given by

$$\langle a(t)\rangle_{\mathrm{q}} = \mathrm{Tr}[\check{A}\check{U}(t,0)\hat{\tilde{\rho}}], \tag{31a}$$

$$\langle a(t_1)a(t_2)\rangle_{\mathrm{q}} = \mathrm{Tr}[\breve{A}\breve{U}(t_2, t_1)\breve{A}\breve{U}(t_1, 0)\hat{\tilde{\rho}}], \tag{31b}$$

$$\langle a(t_1)a(t_2)a(t_3)\rangle_{\mathrm{q}} = \mathrm{Tr}[\breve{A}\breve{U}(t_3, t_2)\breve{A}\breve{U}(t_2, t_1)\breve{A}\breve{U}(t_1, 0)\hat{\tilde{\rho}}],$$
$$\langle a(t_1)\cdots a(t_n)\rangle_{\mathrm{q}} = \mathrm{Tr}[\breve{A}\breve{U}(t_n, t_{n-1})\cdots\breve{A}\breve{U}(t_2, t_1)\breve{A}\breve{U}(t_1, 0)\hat{\tilde{\rho}}], \tag{31c}$$

where $\breve{A}\hat{B} = \{\hat{A}, \hat{B}\}/2$. Note that in (31*a*)–(31*c*), $\hat{\tilde{\rho}} = \hat{\tilde{\rho}}\,(t = 0) = \hat{\rho}$. Equations (31*a*)–(31*c*) provide a more convenient route for the evaluation of these correlation functions.

In the following sections, we apply this general formalism to the two simplest quantum systems: the two-level system and the harmonic oscillator.

4. The two-level system

Consider a two-level system defined by the Hamiltonian

$$\hat{H} = \hbar\omega\hat{\sigma}_x/2 \tag{32}$$

and suppose that the system is in the initial state

$$\hat{\rho}\,(t = 0) = (\hat{1} + \hat{\sigma}_z)/2, \tag{33}$$

where $\hat{\sigma}$ denotes Pauli matrices and $\hat{1}$ is the corresponding unit operator. Left uninterrupted, the system will oscillate between the two eigenstates of $\hat{\sigma}_z$, a process analogous to Rabi oscillations in a harmonically driven system. We focus on the measurement of $\hat{A} = \hat{\sigma}_z$ and denote the measurement outcome by $a\,(t) = \sigma_z\,(t)$. We pose the following questions: can the oscillatory time trace of σ_z be observed? How does the measurement process affect this oscillation? Is the oscillation visible in a single run of an experiment or only as a statistical effect—average over many runs or many copies of the same experiment? The latter question is particularly relevant in light of the growth of activity in single-molecule spectroscopy.

To answer these questions we start by writing the action of $\breve{L}$, (28), in the basis of the Hermitian operators $(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$. In a compact notation it reads

$$\breve{L}(x\hat{\sigma}_x + y\hat{\sigma}_y + z\hat{\sigma}_z) = \omega(y\hat{\sigma}_z - z\hat{\sigma}_y) - 2\lambda(x\hat{\sigma}_x + y\hat{\sigma}_y) \tag{34}$$

and $\breve{L}\hat{1} = 0$. Next, expressing the operation of $\breve{U}\,(t, 0)$ on $\hat{\sigma}_z$ by

$$\hat{\sigma}_z\,(t) = \breve{U}(t, 0)\hat{\sigma}_z = x(t)\hat{\sigma}_x + y(t)\hat{\sigma}_y + z(t)\hat{\sigma}_z \tag{35}$$

and using (34) and (35) in (28) we find $\mathrm{d}x\big/\mathrm{d}t = -2\lambda x$, $\mathrm{d}y\big/\mathrm{d}t = -\,(\omega z + 2\lambda y)$ and $\mathrm{d}z\big/\mathrm{d}t = \omega y$, which, for $z(t = 0) = 1$, $x(0) = y(0) = 0$, yields

$$z(t) = \mathrm{e}^{-\lambda t}[\cos(\Omega t) + \lambda\sin(\Omega t)/\Omega], \;\; y(t) = \mathrm{d}z/\mathrm{d}t, \quad x(t) = 0, \tag{36}$$

where $\Omega = \sqrt{\omega^2 - \lambda^2}$. This allows us to write down the relevant averages (see appendix G), namely

$$\langle\sigma_z(t)\rangle_{\mathrm{q}} = z(t), \;\; \langle\sigma_z(t)\sigma_z(t')\rangle_{\mathrm{q}} = z(|t - t'|). \tag{37}$$

The last line is known in the existing literature only for the stationary case $(t, t' \to \infty)$ in the weak measurement limit $\lambda \to 0$ [17, 24]. It is interesting to note that although the system under consideration is not in a stationary state and in fact evolves irreversibly, this correlation function depends only on the time difference $t' - t$ and remains finite when this difference is constant while both t and t' increase.

Recall that (D.3) and (D.4) imply that $\langle\sigma_z(t)\rangle = \langle\sigma_z(t)\rangle_{\rm q}$, while $\langle\sigma_z(t)\sigma_z(t')\rangle = \langle\sigma_z(t)\sigma_z(t')\rangle_{\rm d} + \langle\sigma_z(t)\sigma_z(t')\rangle_{\rm q}$, which implies $\langle\delta\sigma_z(t)\delta\sigma_z(t')\rangle = \langle\delta\sigma_z(t)\delta\sigma_z(t')\rangle_{\rm d} + \langle\delta\sigma_z(t) \times \delta\sigma_z(t')\rangle_{\rm q}$. In the limit $\lambda \ll \omega$, we see (cf (36) and (37)) clear oscillation of $\langle\sigma_z(t)\rangle$. However, in a single run this signal cannot be distinguished from the noise. Indeed, defining as in (22) $\bar\sigma_z = (1/t_0)\int_0^{t_0} {\rm d}t\sigma_z(t)$, we obviously need to take $t_0 \ll \omega^{-1}$. Therefore

$$\langle(\delta\bar\sigma_z)^2\rangle > \langle((\delta\bar\sigma_z)^2\rangle_{\rm d} = 1/t_0\lambda \gg 1. \tag{38}$$

The large detection noise covers the signal. This implies that Rabi oscillations cannot be seen in a single run.

The above result was obtained in the time domain. We can also ask whether the Rabi oscillation is visible in the frequency domain. This would imply seeing the peak in the Fourier transform

$$\tilde\sigma_z(\nu) = (2/t_0)\int_0^{t_0} {\rm d}t\,\cos(\nu t)\sigma_z(t), \tag{39}$$

where t_0 is a time much longer than the oscillation period, but obviously much smaller than the damping time: $\omega^{-1} \ll t_0 \ll \lambda^{-1}$. From (36) and (37), the peak intensity is $\langle\tilde\sigma_z(\nu = \Omega)\rangle \simeq 1$. On the other hand, under the measurement conditions the white detector noise satisfies

$$\langle(\delta\tilde\sigma_z(\Omega))^2\rangle \geqslant \langle(\delta\tilde\sigma_z(\Omega))^2\rangle_{\rm d} = 1/2t_0\lambda \gg 1, \tag{40}$$

implying that, again, the noise exceeds the signal and a peak in the frequency domain will not be seen. This time, however, the signal-to-noise ratio is not as bad as in the time domain because t_0 can be longer.

We conclude that Rabi oscillation cannot be seen in a single run/copy of the experiment but only in a statistical average. The sample size, i.e. the number of runs/copies needed for this average, is of the order $(t_0\lambda)^{-1}$, where $\lambda^{-1} \gg t_0 \gg \omega^{-1}$ in the frequency domain and $t_0 \ll \omega^{-1} \ll \lambda^{-1}$ in the time domain. In the overdamped regime, $\lambda \gg \omega$, one can see the QZE, discussed below in section 7.

5. The Leggett–Garg-type inequality

The limit $\lambda \to 0$ is consistent with the noninvasive measurement because the backaction vanishes. In this case the negative quasi-probability demonstrates the violation of macrorealism even for a single observable, as shown by Leggett and Garg [28]. In violations of this type, it is essential to subtract the large detection noise, whose uncertainty must always diverge and prevent any real violation [29]. The common confusion about the noninvasiveness condition is caused by the fact that two-time correlations are numerically identical for the quasi-probability in the limit $\lambda \to 0$ and the instant projections (invasive because of collapse) for initial $\hat\rho \sim \hat 1$. The equality still holds in the case of many times if the observable satisfies $\hat A^2 \propto \hat 1$. The analysis above has used second-order correlations that, as stated in (20), are not sensitive to the quasi-probabilistic nature of the distribution. The violation of the well-known Leggett–Garg inequality [28] needs only second-order correlations but requires the additional assumption of bounded observables which is effectively equivalent to higher-order correlations (e.g. the dichotomy $A = \pm 1$ is equivalent to measuring $\langle(A^2 - 1)^2\rangle = 0$, which requires the fourth-order correlator $\langle A^4\rangle$). Without this assumption, the quasi-probabilistic nature is, however, revealed in higher-order correlations. To see this, we take $\hat\rho = \hat 1/2$ and consider the following quantity:

$$X[\sigma] = \sigma_z(0)\sigma_z(\pi/\omega) + \sigma_z(-\pi/2\omega)\sigma_z(\pi/2\omega) + 2. \tag{41}$$

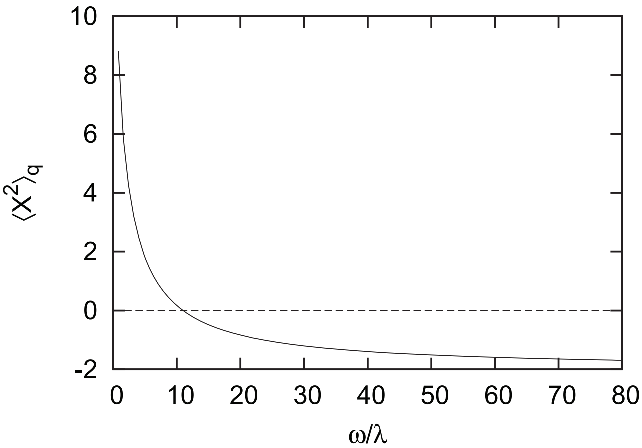

Figure 1. Demonstration of the violation of the Leggett–Garg-type inequality $\langle X^2\rangle_\mathrm{q} > 0$ as a function of measurement strength λ. The function starts from the value 16 for $\lambda \gg \omega$ as expected for the QZE, crosses the classical bound 0 and tends to -2 in the limit of weak measurement $\lambda \ll \omega$.

The fourth-order correlation $\langle X^2\rangle_\mathrm{q}$ is given by

$$\langle X^2\rangle_\mathrm{q} = 6 + \mathrm{e}^{-\lambda\pi/\omega}[1/r^2 + (10 - 1/r^2)\cos(\pi r) + 10\lambda\sin(\pi r)/\omega r], \tag{42}$$

where $r = \sqrt{1-(\lambda/\omega)^2}$. The behavior of $\langle X^2\rangle_\mathrm{q}$ is shown in figure 1. In the limit of strong measurement $\langle X^2\rangle_\mathrm{q} = 16$. The origin is the QZE—the evolution is frozen by the measurement and so $\sigma_z(t)$ does not depend on time, which results in $X = 4$. In the opposite limit of noninvasive measurement $\langle X^2\rangle_\mathrm{q} = -2$ and it crosses zero at $\omega/\lambda \approx 11$. This implies that for a sufficiently small λ the classical inequality $\langle X^2\rangle_\mathrm{q} \geqslant 0$ is violated so the function P_q is not positive definite and as such cannot describe a usual probability. Note, however, that (a) it contains the relevant physical information, discarding the irrelevant detection noise; (b) by itself, it cannot be directly measured, namely correlations such as $\langle X^2\rangle_\mathrm{q}$ are not directly measurable since the real probability is the convolution (14); and (c) the actual detected observable certainly satisfies $\langle X^2\rangle > 0$. However, an independent determination of the detection noise should be experimentally feasible and allows us to find the negativity of $\langle X^2\rangle_\mathrm{q}$ after the noise has been subtracted.

6. The harmonic oscillator

For completeness, we also consider another much studied simple problem—continuous position measurement, $\hat{A} = \hat{x}$, in a system comprising one harmonic oscillator [20], described by the Hamiltonian $\hat{H} = \hat{p}^2/2m + m\omega^2\hat{x}^2/2$. Equations (26) become

$$\begin{aligned} \mathrm{d}\hat{x}_\phi/\mathrm{d}t &= \hat{p}_\phi/m, \\ \mathrm{d}\hat{p}_\phi/\mathrm{d}t &= -m\omega^2\hat{x}_\phi + \hbar\phi(t), \end{aligned} \tag{43}$$

where ϕ represents the zero-centered white Gaussian noise, $\langle\phi(t)\phi(t')\rangle = \lambda\delta(t-t')$. We note in passing that this quantum Langevin equation yields the Fokker–Planck equation for the Wigner function [33]

$$W(x,p)=\int\frac{\mathrm{d}\chi\,\mathrm{d}\xi}{(2\pi)^2}\mathrm{e}^{-\mathrm{i}\xi x-\mathrm{i}\chi p}\mathrm{Tr}\hat{\rho}\,\mathrm{e}^{\mathrm{i}\xi x+\mathrm{i}\chi p} \tag{44}$$

in the form [34]

$$\frac{\partial W(x,p,t)}{\partial t}=m\omega^2 x\frac{\partial W}{\partial p}-\frac{p}{m}\frac{\partial W}{\partial x}+\frac{\lambda\hbar^2}{2}\frac{\partial^2 W}{\partial p^2}. \tag{45}$$

However, in what follows we calculate directly the required correlation functions. Solving (43) we obtain

$$\begin{aligned}\hat{x}_\phi(t)&=\hat{x}(0)\cos\omega t+\frac{\hat{p}(0)}{m\omega}\sin\omega t+\int_0^t\frac{\mathrm{d}t'}{m\omega}\,\sin\omega(t-t')\,\hbar\phi(t),\\ \hat{p}_\phi(t)&=\hat{p}(0)\cos\omega t-m\omega\hat{x}(0)\sin\omega t+\int_0^t\mathrm{d}t'\,\cos\omega(t-t')\,\hbar\phi(t').\end{aligned} \tag{46}$$

This implies that the oscillations of the average position are undamped,

$$\langle x(t)\rangle=\langle x(0)\rangle\cos(\omega t)+(m\omega)^{-1}\langle p(0)\rangle\sin(\omega t) \tag{47}$$

independently of the detection strength.

Turning to the noise term we first note that, as before, the detector noise combines additively with the correlation functions obtained from (46). The latter take the form

$$\langle\delta x(t)\delta x(t')\rangle_{\mathrm{q}}=\langle\delta x(t)\delta x(t')\rangle_0+f_\lambda(t,t'), \tag{48}$$

where $\langle\delta x(t)\delta x(t')\rangle_0$ is the free correlation function obtained in the limit $\lambda\to 0$ or equivalently $\phi\to 0$, that is, by ignoring the last (noise) terms on the RHS of (46),

$$\begin{aligned}\langle\delta x(t)\delta x(t')\rangle_0&=\langle\delta x(0)\delta x(0)\rangle\cos\omega t\cos\omega t'\\ &+\langle\delta x(0)\delta p(0)\rangle_{\mathrm{W}}(m\omega)^{-1}\sin\omega(t+t')+\langle\delta p(0)\delta p(0)\rangle(m\omega)^{-2}\sin\omega t\sin\omega t'\end{aligned} \tag{49}$$

with the Wigner-ordered average $\langle 2xp\rangle_{\mathrm{W}}=\mathrm{Tr}[\hat{\rho}\{\hat{x},\hat{p}\}]$, and where $f_\lambda(t,t')$ is the correlation function associated with the noise terms in (46),

$$f_\lambda(t,t')=\frac{\lambda\hbar^2}{2(m\omega)^2}[\min(t,t')\cos\omega(t-t')+(\sin\omega|t-t'|-\sin\omega(t+t'))/2\omega]. \tag{50}$$

This measurement-induced correlation function represents the backaction effect of the measuring process. It depends on the detector strength and the parameters of the dynamics but not on the initial state of the oscillator. Moreover, because of the Gaussian nature of ϕ, it contributes solely to the second cumulant $\langle\langle x(t)x(t')\rangle\rangle$, leaving all the others unaffected. As expected, it vanishes in the limit $\lambda\to 0$. However, the most striking feature in (50) is the growth of noise with time, as expressed by the first term in (50).

In analogy to the two-level system we discuss the behavior of the short-time ($t\ll\omega^{-1}$) average $\bar{x}$ and the long-time ($t\gg\omega^{-1}$) Fourier transform $\tilde{x}$, defined by the analogues of (22) and (39), respectively. In both limits we are now free to choose $t_0\lambda$ because, in contrast to the two-level case, the averaged oscillations, (47), are not damped.

Consider first the time-domain observation. For $t_0\lambda\ll 1$ the uncertainty of $\bar{x}$ is determined by the detection noise as in (38), while in the opposite limit it will be dominated by the backaction (50). In either case the noise exceeds the signal.

In the frequency domain, for $\tilde{x}(\omega) = 2\int_0^{t_0} \mathrm{d}t \cos(\omega t)x(t)/t_0$, we obtain, using (47)–(50), the peak signal

$$\langle \tilde{x}(\omega)\rangle = \langle x(0)\rangle , \tag{51}$$

and the intrinsic and backaction noise components

$$\langle(\delta\tilde{x}(\omega))^2\rangle_{\mathrm{q}} \simeq \langle(\delta x(0))^2\rangle + \frac{\lambda t_0 \hbar^2}{6\,(m\omega)^2} \tag{52}$$

to which we need to add the detector noise $(2t_0\lambda)^{-1}$. The total uncertainty originating from the detector satisfies

$$\frac{\lambda t_0 \hbar^2}{6\,(m\omega)^2} + \frac{1}{2t_0\lambda} \geqslant \frac{\hbar}{\sqrt{3}m\omega} \tag{53}$$

with the lower bound (obtained as the minimum of the Lhs with respect to $t_0\lambda$) independent of λ and t_0. Obviously, $\langle x(0)\rangle$ can be chosen large enough for the signal to dominate the noise at intermediate times, but the noise will always exceed the signal at long enough times. As always, the signal-to-noise ratio can be improved by repeated measurements.

The fact that the backaction contribution (50) to the noise grows with time reflects the continuous pumping of energy to the system affected by the measurement process [35]. This does not happen in the two-level system because of its bounded spectrum; still also in that system the temperature grows to infinity ($\hat{\rho}(t) \to (1/2)\hat{1}$) as implied by equations (35)–(37). This unlimited growth can be avoided by assuming that the measurement process also involves some friction [21, 34]. Indeed, measurement, even classical, means extraction of information out of the system, so that without compensating for friction its entropy must increase and so does the temperature.

7. The quantum Zeno effect

For completeness, we show now how the QZE emerges within the present formalism. So far we have focused on weak measurements, represented by small λ. The opposite limit, $\lambda \to \infty$, represents the strong measurement case. In systems characterized by a single timescale ω^{-1}, strong and weak measurements are quantified by the inequalities $\lambda \gg \omega$ and $\lambda \ll \omega$, respectively.

Consider the two-level system discussed in section 4. For $\lambda > \omega$ its dynamics is given by the overdamped analogue of equation (36), $\Omega = \mathrm{i}\sqrt{\lambda^2 - \omega^2}$. In the extreme strong measurement case, $\lambda \gg \omega$, $z(t) \sim \mathrm{e}^{-\omega^2 t/2\lambda}$ and the decay slows down as $\lambda \to \infty$ [14, 36]. This corresponds to the QZE where the system is almost frozen by the measurement, reaching its equilibrium state $z = 0$ only on the timescale $t \sim \lambda/\omega^2$.

For a position measurement in the harmonic oscillator case, we have seen, equation (47), that the average position oscillates regardless of the strength of the measurement. This implies that the Zeno effect is absent in this system, as is well known [14]. On the other hand, for any measurement strength, the detector-induced backaction noise, equation (50), increases without bound at long times at a rate that increases with λ. Already for short times we get $f_\lambda(t, t) \simeq \lambda\hbar^2/3\,\mathrm{m}^2$, and backaction adds fast diffusion in the phase space. This is somewhat analogous to the anti-Zeno effect [27].

8. Conclusions

Gaussian POVMs, represented here by the Kraus operators, were used in this paper to formalize the description of weak measurements. A path integral representation of continuous weak measurement described in this way leads directly to an analysis of backaction noise in terms of stochastic evolution equations. The average signal and the associated noise were obtained in terms of moments and time correlation functions of the measured quantity.

In particular, the noise was shown to be an additive combination of a term characteristic of the measurement alone (detector noise) and terms associated with the system, which in turn include contributions from the intrinsic quantum mechanical uncertainty in the system and from backaction effects from the measurement process. A transparent representation of this stochastic evolution was obtained by separating it into a process characteristic only of the weak measurement and another representing the quantum uncertainty intrinsic to the system as well as that arising from the measurement backaction. This yields the noise as an additive combination of the corresponding contributions, while the total probability is found to be the convolution of white Gaussian detections noise and the intrinsic system's quasi-probability. The quasi-probability can be negative although the negativity is not visible at the level of second-order correlations due to weak positivity. The general formalism was applied to two simple problems: continuous monitoring of the level population in a two-level system and continuous measurement of the position of a harmonic oscillator. For both systems we have established limits on the possibility of observing oscillatory motion in a single run of an experiment. The negativity property of the quasi-probability can be demonstrated in the two-level system using fourth-order correlations. In this way, we have constructed a Leggett–Garg-type inequality without the assumption of dichotomy or boundedness of the variable.

We observe that the QZE occurs when both the Hamiltonian and the observable can be represented in finite-dimensional Hilbert space. When the space is infinite or continuous and both the Hamiltonian and the observable have no finite-dimensional representation, the dynamics will not always be able to 'pin down' the state and consequently the dynamics may get diffusive. Establishing criteria for the occurrence or absence of the QZE in realistic systems continues to be an intriguing and challenging issue.

Acknowledgments

We are grateful to J Audretsch and Y Aharonov for fruitful discussions. AN acknowledges support from the European Science Council (FP7/2007–2013, grant agreement no. 226628), the Israel–Niedersachsen Research Fund and the Israel Science Foundation. He also thanks the Alexander von Humboldt Foundation and SFB 767 for sponsoring his visit to the University of Konstanz. WB and AB acknowledge support from the DFG through SFB 767 and SP 1285. All the authors also thank the Kurt Lion Foundation for supporting this work.

Appendix A

Here we derive (5). Using (2), a succession of time evolutions and measurements in the interval $(0, t)$ reads

$$\hat{K}(\{a_j\}) = (2\bar{\lambda}/\pi)^{N/4} e^{-(i/\hbar)\hat{H}(t_{N+1}-t_N)} e^{-\bar{\lambda}(a_N-\hat{A})^2} \ldots \\ e^{-(i/\hbar)\hat{H}(t_3-t_2)} e^{-\bar{\lambda}(a_2-\hat{A})^2} e^{-(i/\hbar)\hat{H}(t_2-t_1)} e^{-\bar{\lambda}(a_1-\hat{A})^2} e^{-(i/\hbar)\hat{H}t_1}. \tag{A.1}$$

Putting $\bar{\lambda} = \lambda\Delta t$ and $t_j - t_{j-1} = t_1 = \Delta t$ and using $\Delta t \to 0$ leads to

$$\hat{K}(\{a_j\}) = (2\lambda\Delta t/\pi)^{N/4} \prod_{j=1}^{N} \mathrm{e}^{[-(\mathrm{i}/\hbar)\hat{H} - \lambda(a_j - \hat{A})^2]\Delta t} \mathrm{e}^{-(i/\hbar)H\Delta t} \tag{A.2}$$

and, for $\Delta t \to 0$,

$$\hat{K}[a(t)] = C\mathcal{T}\mathrm{e}^{\int_0^t [-(\mathrm{i}/\hbar)\hat{H} - \lambda(a(t) - \hat{A})^2]\mathrm{d}t}. \tag{A.3}$$

Alternatively, using $\hat{A}(t_j) = \mathrm{e}^{(\mathrm{i}/\hbar)\hat{H}t_j} \hat{A}\, \mathrm{e}^{-(\mathrm{i}/\hbar)\hat{H}t_j}$ yields

$$\hat{K}(\{a_j\}) = (2\lambda\Delta t/\pi)^{N/4}\, \mathrm{e}^{-(\mathrm{i}/\hbar)\hat{H}t_{N+1}} \prod_{j=1}^{N} \mathrm{e}^{[-\lambda(a_j - \hat{A}(t_j))^2]\Delta t} \tag{A.4}$$

and in the continuum limit

$$K\,[a\,(t)] = C\,\mathrm{e}^{-(\mathrm{i}/\hbar)Ht}\mathcal{T}\mathrm{e}^{-\lambda\int_0^t \left(a(t) - \hat{A}(t)\right)^2 \mathrm{d}t}. \tag{A.5}$$

In (A.3) and (A.5), C are normalization factors.

Appendix B

Here we prove (10). Start from $\mathcal{T}\exp\int \mathrm{i}(\chi(t)/2 + \phi(t))\hat{A}(t)\mathrm{d}t$ and discretize to obtain

$$\begin{aligned}
\mathcal{T}\mathrm{e}^{\mathrm{i}\int(\chi(t)/2+\phi(t))\hat{A}(t)\mathrm{d}t} &= \mathcal{T}\mathrm{e}^{\mathrm{i}\Delta t\sum_j(\chi(t_j)/2+\phi(t_j))\hat{A}(t_j)} \\
&= \left\{t_j = j\Delta t;\; j = 1, \ldots, N\right\} \\
&\quad \times \mathrm{e}^{\mathrm{i}\Delta t(\chi(t_N)/2+\phi(t_N))\hat{A}(t_N)} \mathrm{e}^{\mathrm{i}\Delta t(\chi(t_{N-1})/2+\phi(t_{N-1}))\hat{A}(t_{N-1})} \ldots \mathrm{e}^{\mathrm{i}\Delta t(\chi(t_1)/2+\phi(t_1))\hat{A}(t_1)} \\
&= \mathrm{e}^{(\mathrm{i}/\hbar)Ht_N} \mathrm{e}^{\mathrm{i}\Delta t(\chi(t_N)/2+\phi(t_N))\hat{A}} \mathrm{e}^{-(rmi/\hbar)Ht_N} \\
&\quad \times \mathrm{e}^{(\mathrm{i}/\hbar)Ht_{N-1}} \mathrm{e}^{\mathrm{i}\Delta t(\chi(t_{N-1})/2+\phi(t_{N-1}))\hat{A}} \mathrm{e}^{-(\mathrm{i}/\hbar)Ht_{N-1}} \\
&\quad \cdots\, \mathrm{e}^{(\mathrm{i}/\hbar)Ht_1} \mathrm{e}^{\mathrm{i}\Delta t(\chi(t_1)/2+\phi(t_1))\hat{A}} \mathrm{e}^{-(\mathrm{i}/\hbar)Ht_1}.
\end{aligned} \tag{B.1}$$

Next replace

$$\mathrm{e}^{\mathrm{i}\Delta t(\chi(t_j)/2+\phi(t_j))\hat{A}} \to \mathrm{e}^{\mathrm{i}\Delta t\hat{A}\chi(t_j)/2} \mathrm{e}^{\mathrm{i}\hat{A}\phi(t_j)\left(t_j - t_{j-1}\right)} \tag{B.2}$$

for $j = 1, \ldots, N$ and define $\hat{H}_\phi\,(t) = \hat{H} - \hbar\phi\,(t)\,\hat{A}$, $\hat{A}_\phi(t) = \tilde{\mathcal{T}}\mathrm{e}^{(\mathrm{i}/\hbar)\int 0^t \hat{H}_\phi(t')\mathrm{d}t'} \hat{A} \times \mathcal{T}\mathrm{e}^{-(\mathrm{i}/\hbar)\int_0^t \hat{H}_\phi(t')\mathrm{d}t'}$ and discretize it again,

$$\hat{A}_\phi(t_k) = \tilde{\mathcal{T}} \prod_{j\leqslant k} \mathrm{e}^{(\mathrm{i}/\hbar)\hat{H}_\phi(t_j)\Delta t} \hat{A}\mathcal{T} \prod_{j\leqslant k} \mathrm{e}^{-(\mathrm{i}/\hbar)\hat{H}_\phi(t_j)\Delta t}, \tag{B.3}$$

to obtain

$$\mathrm{e}^{\mathrm{i}t_N\phi(t_N)}\mathcal{T}\mathrm{e}^{\mathrm{i}\Delta t\sum_{j=1}^N \chi(t_j)\hat{A}_\phi(t_j)/2} \to \mathrm{e}^{\mathrm{i}t\phi(t)}\mathcal{T}\mathrm{e}^{\mathrm{i}\int_0^t(\chi(t')/2)\hat{A}_\phi(t')\mathrm{d}t'} \tag{B.4}$$

from which follows (10).

Appendix C

Here we derive equations (19*a*)–(19*c*). Start from (13) and take its functional derivatives

$$\begin{aligned}\langle a\,(t)\rangle_{\rm b} &= \int Da\; a\,(t)\; P_{\rm b}[a] = \frac{1}{i}\left(\frac{\delta {\rm e}^{S_{\rm b}[\chi]}}{\delta\chi\,(t)}\right)_{\chi(t)=0}\\ &= \int D\phi\,{\rm e}^{-\int {\rm d}t\phi^2(t)/2\lambda}{\rm Tr}\,\mathcal{T}(\hat{A}_\phi(t)\hat{\rho}+\hat{\rho}\hat{A}_\phi(t))/2\\ &= \int D\phi\,{\rm e}^{-\int {\rm d}t\phi^2(t)/2\lambda}{\rm Tr}(\hat{A}_\phi(t)\hat{\rho}),\end{aligned}\tag{C.1}$$

which is (19*a*).

$$\begin{aligned}\langle a(t)a(t')\rangle_{\rm b} &= \int Da\; a\,(t)\,a\left(t'\right)P_{\rm b}[a] = -\left(\frac{\delta^2{\rm e}^{S_{\rm b}[\chi]}}{\delta\chi\,(t)\,\delta\chi\,(t')}\right)_{\chi(t)=0}\\ &= \frac{1}{4}\int D\phi\,{\rm e}^{-\int {\rm d}t\phi^2(t)/2\lambda}{\rm Tr}\{\hat{A}_\phi(t'),\{\hat{A}_\phi(t),\hat{\rho}\}\}.\end{aligned}\tag{C.2}$$

Time ordering implies that for $t' > t$, t' will be placed in the outer commutator; however, the last expression is equal to

$$\frac{1}{2}\int D\phi\,{\rm e}^{-\int {\rm d}t\phi^2(t)/2\lambda}{\rm Tr}\,\{\hat{A}_\phi(t),\hat{A}_\phi(t')\}\hat{\rho},\tag{C.3}$$

which does not depend on the operator ordering. Higher moments are obtained in the same way.

Appendix D

Here we consider the case $[\hat{A},\hat{H}]=0$ and $[\hat{\rho},\hat{A}]=0$. When the observable $\hat{A}$ commutes with the Hamiltonian, $\hat{A}_\phi\,(t)=\hat{A}$, so the trace in (13) becomes independent of ϕ. Using $\int D\phi\,{\rm e}^{-\int {\rm d}t\phi^2(t)/2\lambda}=1$ it follows that ${\rm e}^{S_{\rm b}[\chi]}$ and $P_{\rm b}[a]$, (16), do not depend on λ. This implies that the evolution associated with the backaction effect is deterministic and the only source of noise is the detector. To see the implication of this on the moments consider the moment, generating function (cf (12) and (13))

$${\rm e}^{S[\chi]}={\rm e}^{S_{\rm d}[\chi]}{\rm e}^{S_{\rm b}[\chi]}={\rm e}^{-\int {\rm d}t\chi^2(t)/8\lambda}\int D\phi\,{\rm e}^{-\int {\rm d}t\phi^2(t)/2\lambda}{\rm Tr}\,\mathcal{T}{\rm e}^{{\rm i}\int\chi(t)\hat{A}_\phi(t){\rm d}t/2}\;\hat{\rho}\;\tilde{\mathcal{T}}{\rm e}^{{\rm i}\int\chi(t)\hat{A}_\phi(t){\rm d}t/2}.\tag{D.1}$$

For the imposed initial conditions, this becomes

$${\rm e}^{S[\chi]}={\rm e}^{-\int {\rm d}t\chi^2(t)/8\lambda}\,({\rm e}^{{\rm i}a\int\chi(t){\rm d}t/2})^2.\tag{D.2}$$

The first moment satisfies

$$\langle a\rangle=\left(\frac{\delta}{{\rm i}\delta\chi\,(t)}{\rm e}^{S_{\rm d}[\chi]+S_{\rm b}[\chi]}\right)_{\chi(t)=0}=\left(\frac{\delta}{{\rm i}\delta\chi\,(t)}{\rm e}^{S_{\rm b}[\chi]}\right)_{\chi(t)=0}=a_0,\tag{D.3}$$

which implies also $\langle \bar{a} \rangle = a$ when used in (22). It is easy to realize that the second moment satisfies

$$\langle a(t)a(t')\rangle = -\left(\frac{\delta^2}{\delta\chi\,(t)\,\delta\chi\,(t')}\mathrm{e}^{S_\mathrm{d}[\chi]+S_\mathrm{b}[\chi]}\right)_{\chi(t)=0}$$
$$= -\left(\frac{\delta^2}{\delta\chi\,(t)\,\delta\chi\,(t')}\mathrm{e}^{S_\mathrm{d}[\chi]}\right)_{\chi(t)=0} - \left(\frac{\delta^2}{\delta\chi\,(t)\,\delta\chi\,(t')}\mathrm{e}^{S_\mathrm{b}[\chi]}\right)_{\chi(t)=0}. \tag{D.4}$$

The second term yields a^2, so $\langle \delta a\,(t)\,\delta a\,(t')\rangle = \langle \delta a\,(t)\,\delta a\,(t')\rangle - a^2$ is determined just by the Gaussian detector noise that in the case of (22) results in

$$\langle (\delta\bar{a})^2\rangle = 1/\,(4\lambda t_0)\,. \tag{D.5}$$

Appendix E

Here we derive the master equation (28). Following the steps that lead to (9) but without the trace, we find that

$$\hat{\rho}[a] = \mathrm{e}^{-(\mathrm{i}/\hbar)\hat{H}t}\int D\phi\,\mathrm{e}^{-\int \mathrm{d}t\phi^2(t)/2\lambda}\int D\chi\,\mathrm{e}^{-\int \mathrm{d}t\chi^2(t)/8\lambda}\mathrm{e}^{-\int \mathrm{i}\chi(t)a(t)\mathrm{d}t}$$
$$\times\mathcal{T}\mathrm{e}^{\int \mathrm{i}(\chi(t)/2+\phi(t))\hat{A}(t)\mathrm{d}t}\;\hat{\rho}\;\tilde{\mathcal{T}}\mathrm{e}^{\int \mathrm{i}(\chi(t)/2-\phi(t))\hat{A}(t)\mathrm{d}t}\mathrm{e}^{(\mathrm{i}/\hbar)\hat{H}t} \tag{E.1}$$

and

$$\hat{\bar{\rho}}\,(t) \equiv \int Da\hat{\rho}[a]$$
$$= \mathrm{e}^{-(\mathrm{i}/\hbar)\hat{H}t}\int D\phi\,\mathrm{e}^{-\int \mathrm{d}t\phi^2(t)/2\lambda}\mathcal{T}\mathrm{e}^{\mathrm{i}\int \phi(t)\hat{A}(t)\mathrm{d}t}\;\hat{\rho}\;\tilde{\mathcal{T}}\mathrm{e}^{-\mathrm{i}\int \phi(t)\hat{A}(t)\mathrm{d}t}\mathrm{e}^{(\mathrm{i}/\hbar)\hat{H}t}. \tag{E.2}$$

In what follows, we will use the incremental propagation version of this equation:

$$\hat{\bar{\rho}}\,(t+\Delta t) = \mathrm{e}^{-(\mathrm{i}/\hbar)\hat{H}(t+\Delta t)}\int D\phi\,\mathrm{e}^{-\int_t^{t+\Delta t}\mathrm{d}t\phi^2(t')/2\lambda}$$
$$\times\mathcal{T}\mathrm{e}^{\mathrm{i}\int_t^{t+\Delta t}\phi(t')\hat{A}(t')\mathrm{d}t}\;\hat{\bar{\rho}}\,(t)\;\tilde{\mathcal{T}}\mathrm{e}^{-\mathrm{i}\int_t^{t+\Delta t}\phi(t')\hat{A}(t')\mathrm{d}t}\mathrm{e}^{(\mathrm{i}/\hbar)\hat{H}(t+\Delta t)}. \tag{E.3}$$

Next use

$$\mathcal{T}\mathrm{e}^{\mathrm{i}\int_t^{t+\Delta t}\phi(t')\hat{A}(t')\mathrm{d}t} = \mathcal{T}\exp(\mathrm{i}\Delta t\phi(t)\mathrm{e}^{(\mathrm{i}/\hbar)\hat{H}t}\hat{A}\mathrm{e}^{-(\mathrm{i}/\hbar)\hat{H}})$$
$$= \prod_j \mathrm{e}^{(\mathrm{i}/\hbar)\hat{H}t_j}\mathrm{e}^{\mathrm{i}\Delta t\phi(t)\hat{A}}\mathrm{e}^{-(\mathrm{i}/\hbar)\hat{H}t_j}$$
$$= \mathrm{e}^{(\mathrm{i}/\hbar)\hat{H}(t+\Delta t)}\mathrm{e}^{\int_t^{t+\Delta t}(\mathrm{i}/\hbar)[\phi(t')\hat{A}-\hat{H}]\mathrm{d}t'}\mathrm{e}^{-(\mathrm{i}/\hbar)\hat{H}t} \tag{E.4}$$

to rewrite (E.3) in the form

$$\hat{\bar{\rho}}\,(t+\Delta t) = \int D\phi\,\mathrm{e}^{-\int_t^{t+\Delta t}\mathrm{d}t\phi^2(t)/2\lambda}\,\mathrm{e}^{\mathrm{i}\int_t^{t+\Delta t}(\phi(t')\hat{A}-\hat{H})\mathrm{d}t}\;\hat{\bar{\rho}}\,(t)\;\mathrm{e}^{-\mathrm{i}\int_t^{t+\Delta t}(\phi(t')\hat{A}-\hat{H})\mathrm{d}t}. \tag{E.5}$$

We next expand the Rhs of (E.5), keeping only terms that can contribute to order $O(\Delta t)$. To this end, we use

$$e^{\pm i\int_t^{t+\Delta t} dt\,(\phi(t)\hat{A}-\hat{H}/\hbar)} = 1 \pm i\hat{A}\int_t^{t+\Delta t}\phi(t)dt \mp i(\hat{H}/\hbar)\Delta t$$
$$-\hat{A}^2\int_t^{t+\Delta t}\int_t^{t+\Delta t} dt\,dt'\phi(t)\phi(t')/2. \tag{E.6}$$

This leads, using $\langle\phi\rangle = 0$ and $\langle\phi(t)\phi(t')\rangle = \lambda\delta(t-t')$, to

$$\hat{\tilde{\rho}}(t+\Delta t) = \hat{\tilde{\rho}}(t) - \Delta t[\hat{H},\hat{\tilde{\rho}}(t)](i/\hbar) - i[\hat{A},\hat{\tilde{\rho}}(t)]\int_t^{t+\Delta t}\langle\phi(t)\rangle\,dt$$
$$+\int_t^{t+\Delta t}\langle\phi(t)\phi(t')\rangle(\hat{A}\hat{\tilde{\rho}}(t)\hat{A} - \{\hat{A}^2,\hat{\tilde{\rho}}(t)\}/2)$$
$$= \hat{\rho}(t) - \Delta t[\hat{H},\hat{\tilde{\rho}}(t)]\,(i/\hbar) - \lambda\Delta t\{\hat{A}^2,\hat{\tilde{\rho}}(t)\}/2 + \lambda\Delta t\hat{A}\hat{\tilde{\rho}}(t)\hat{A}, \tag{E.7}$$

which yields

$$\frac{d\hat{\tilde{\rho}}(t)}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\tilde{\rho}}(t)] - \lambda\{\hat{A}^2,\hat{\tilde{\rho}}(t)\}/2 + \lambda\hat{A}\hat{\tilde{\rho}}(t)\hat{A}$$
$$= [\hat{H},\hat{\tilde{\rho}}(t)]/i\hbar - \lambda[\hat{A},[\hat{A},\hat{\tilde{\rho}}(t)]]/2. \tag{E.8}$$

Appendix F

Here we derive (31*a*)–(31*c*). We will demonstrate the derivation of the two-time correlation function, (31*b*). We start from (19*b*) in the form (C.2) and use the cyclic permutation property of the trace together with identities such as

$$\mathcal{T}e^{(i/\hbar)\int_0^{t'} d\tau\,\hat{H}_\phi(\tau)}\tilde{\mathcal{T}}e^{(i/\hbar)\int_0^{t} d\tau\,\hat{H}_\phi(\tau)} = \mathcal{T}e^{(i/\hbar)\int_t^{t'} d\tau\,\hat{H}_\phi(\tau)}\ (t'>t) \tag{F.1}$$

to obtain

$$\langle a(t)a(t')\rangle_b = \frac{1}{2}\int D_\lambda\phi\,\mathrm{Tr}\left[\hat{A}\tilde{\mathcal{T}}e^{\int_t^{t'}\hat{H}_\phi(s)ds/i\hbar}\left\{\hat{A},\tilde{\mathcal{T}}e^{\int_0^t\hat{H}_\phi(s)ds/i\hbar}\hat{\rho}\mathcal{T}e^{\int_0^t i\hat{H}_\phi(s)ds/\hbar}\right\}\mathcal{T}e^{\int_t^{t'} i\hat{H}_\phi(s)ds/\hbar}\right], \tag{F.2}$$

where $\hat{H}_\phi(t) = \hat{H} - \hbar\phi(t)\hat{A}$. The functional integral $\int D_\lambda\phi$ can be divided into a product of integrals performed over trajectories $\phi(t)$ between 0 and t and between t and t'. The former operates only on the ϕ-dependent expression in the anticommutator brackets, yielding $\int D_\lambda\phi\,\tilde{\mathcal{T}}e^{-\int_0^t i\hat{H}_\phi(s)ds/\hbar}\hat{\rho}\mathcal{T}e^{\int_0^t i\hat{H}_\phi(s)ds/\hbar}$, which satisfies

$$\int D_\lambda\phi\,\tilde{\mathcal{T}}e^{-\int_0^t i\hat{H}_\phi(s)ds/\hbar}\hat{\rho}\mathcal{T}e^{\int_0^t i\hat{H}_\phi(s)ds/\hbar} = \breve{U}(t,0)\,\hat{\rho} = \hat{\tilde{\rho}}(t) \tag{F.3}$$

(equation (F.3) is equivalent to (E.5), generalized for finite time evolution). Equation (F.2) becomes

$$\langle a(t)a(t')\rangle_b = \int D_\lambda\phi\mathrm{Tr}\left[\hat{A}\tilde{\mathcal{T}}e^{-i\int_t^{t'}\hat{H}_\phi(s)ds/\hbar}\breve{A}\breve{U}(t,0)\,\hat{\rho}\mathcal{T}e^{i\int_t^{t'}\hat{H}_\phi(s)ds/\hbar}\right]. \tag{F.4}$$

Using (F.3) again, now in the form $\int D_\lambda\phi\,\tilde{\mathcal{T}}e^{-\int_t^{t'} i\hat{H}_\phi(s)ds/\hbar}\hat{x}(t)\,\mathcal{T}e^{\int_t^{t'} i\hat{H}_\phi(s)ds/\hbar} = \breve{U}(t',t)\,\hat{x}(t)$, leads to (31*b*). Equation (31*c*) is verified analogously.

Appendix G

Here we prove (37). From (31*a*), (33) and (35), using also $\breve{U}\hat{1} = 1$ (since $\breve{L}\hat{1} = 0$) and $\{\hat{\sigma}_j, \hat{\sigma}_k\} = 2\delta_{jk}\hat{1}$, we obtain

$$\langle\sigma_z(t)\rangle_{\rm b} = \frac{1}{2}{\rm Tr}[\hat{\sigma}_z(1+\hat{\sigma}_z(t))] = \frac{z(t)}{2}{\rm Tr}(\hat{\sigma}_z^2) = z(t). \tag{G.1}$$

Next, from (31*b*) and (33)

$$\langle\sigma_z(t)\sigma_z(t')\rangle_{\rm b} = \frac{1}{4}{\rm Tr}[\hat{\sigma}_z\breve{U}(t',t)\{\hat{\sigma}_z, (\hat{1}+\hat{\sigma}_z(t))\}] \tag{G.2}$$

for $t' > t$. Using (35) gives $\{\hat{\sigma}_z, (\hat{1}+\sigma_z(t))/2\} = \hat{\sigma}_z + z(t)\hat{1}$. Finally,

$$\begin{aligned}\langle\sigma_z(t)\sigma_z(t')\rangle_{\rm b} &= \frac{1}{2}{\rm Tr}[\hat{\sigma}_z\breve{U}(t',t)(\hat{\sigma}_z+z(t)\hat{1})]\\ &= \frac{1}{2}{\rm Tr}[\hat{\sigma}_z(\hat{\sigma}_z(t'-t)+z(t)\hat{1})] = z(t'-t).\end{aligned} \tag{G.3}$$

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